Advanced Mathematics Support Programme®

Reasoning and Proof
Introducing Reasoning, Logic and Proof

Aim	One of the overarching themes in the AS and A levels from 2017 onwards is Mathematical Argument, Logic and Proof. The description in the GCE AS and A level subject content states that students should be able to construct and present mathematical arguments through ● appropriate use of diagrams ● sketching graphs ● logical deduction ● precise statements involving correct use of symbols and connecting language o this includes the terms constant, coefficient, expression, equation, function, identity, index, term and variable They are also expected to comprehend and critique mathematical arguments, proofs and justifications of methods and formulae, including those relating to applications of mathematics. This activity aims to introduce students to some of the basic ideas involved in reasoning, logic and proof at A level, in particular the phrases "true", "false", "always true", "always false", "counter example" and "proof."
Activity	Teacher puts the following pair of statements on the board: A: x is an even number B: x is a multiple of 4 The students are asked • If you know that statement A is true – we have an even number, does that mean that statement B is always true, sometimes true or never true? When the students settle on sometimes true, the teacher can ask "can you tell me how you know it isn't *always* true?" Hopefully a student will say that it is because e.g. 6 is an even number that is not a multiple of 4 so not all even numbers are multiples of 4. The teacher surmises that as there exists an even number that is not a multiple of 4, it follows that statement A being true does not always mean that statement B is true. The teacher introduces the phrase "counter example" and explains that it is a good way of proving that something is not always true. The students are then asked • If you know that statement B is true – we have a multiple of 4, does that mean that statement A is always true, sometimes true or never true?

	The teacher brings out from the class that you can't find a multiple of 4 that is not even and so the statement is always true. The teacher then gets the students to think why it is always true logically and then shows this simple proof: A multiple of 4 can be written $4n$, a multiple of 2 can be written $2m$, where m and n are integers. $4n = 2 \times (2n)$ $= 2 \times m$ Word explanations are acceptable too: It's always 2 times an even number etc. etc. The teacher then does something similar with two new statements A: $x > 45$ B: $x > 44.7$ If A is true, B is always true. Proof $45 > 44.7$. If B is true A is not necessarily true. $x = 44.8$ is a counter example. The teacher then does one much more difficult one A: $\sin x > 0$ B: $\tan x > 0$ This one lets the teacher show how graphs can be useful - the students can search for counter examples on their calculators If A is true, B is not always true $\sin 120°$ and $\tan 120°$ are a counter example. If B is true, A is not always true $\sin 210°$ and $\tan 210°$ are a counter example. The students then work with pairs of statements searching for counter-examples and explaining their thinking. Once the students have had a reasonable time to look at the statements and come up with counter examples, the teacher summarises some of the better responses and examples.
The statements used in the video are shown on the next page.	

For each pair of statements

a) If A is true, is B always or sometimes true?
b) If B is true, is A always or sometimes true?

Use counter-examples to show cases where one statement being true doesn't necessarily mean that the other is true.

	Statement A	Statement B
1	$a = b$	$a + c = b + c$
2	$x^2 = 25$	$x = 5$
3	$\|x\| > 3$	$x > 3$
4	$a = b$	$ac = bc$
5	$x^2 < 9$	$x < 3$
6	$y < 0$	$xy < 0$
7	a, b and c are consecutive numbers	abc is a multiple of 3
8	p is a prime number greater than 3	p is one more or one less than a multiple of 6
9	a, b and c are consecutive numbers	abc is a multiple of 6
10	$u_1 = 1, u_{n+1} = u_n + n + 1$	$u_n = \frac{1}{2}n(n+1)$