Advanced Mathematics Support Programme

# Algebraic proof

## A formula for prime numbers

What is a prime number? Any number that has exactly two factors

Write down the first 12 prime numbers. 2 3 5 7 11 13 17 19 23 27 29 31

Here is a formula for the prime numbers, where n is a whole number (1,2,3 ….) and P is a prime number

$P = 6n - 1$ and $P = 6n + 1$

So, when n = 1 P = 5 and 7 **5 and 7 are both prime numbers**

Check if the formula works when

| | | |
|---|---|---|
| n = 2 | P = 11 and 13 | 11 & 13 are both prime |
| n = 3 | P = 17 and 19 | 17 & 19 are both prime |
| n = 4 | P = 23 and 25 | 25 is not prime |
| n = 5 | P = 29 and 31 | 29 & 31 are both prime |

Did you get a prime number every time? No

If you found one value of n that doesn't give you a prime number, then the formula doesn't work.
Often, it is easier to prove a formula isn't true than to prove that it is true.

Here is another formula for prime numbers:

$$P = n^2 + n + 41$$

Choose some values for n. Does it always work? How do you know?
(to help you, the first 75 prime numbers are written on the last page)

The last page shows results for n = 1 to 60
The first value of n which does not yield a prime number is 40 which gives 1681 (41 x 41)
It's unlikely that students will discover a value that doesn't work, so this is a good opportunity for students to see that lots of examples that work is not evidence that it is always right.
If it is practical, this is also a good opportunity for students to use a spreadsheet to generate the repetitive calculations (they will of course need a longer list of prime numbers to compare with).

***To prove a rule algebraically, you have to be able to prove it works every time, for every value of n.***

***One example where it does not work is enough to say rule isn't true***

***Lots of examples where the rule does work is not enough to prove the rule is true***

***You need a way to show that the rule works for every possible value of n***

Lots of people have tried to find a formula for prime numbers. Some of these formulae look really complicated and seem to work for a lot of values of n, but so far no-one has found one that works for every value of n. So, if you have nothing to do next weekend…. (but don't waste too much time on it!)

In GCSE maths you may be asked to prove something a bit simpler, like this:

**Prove that the difference between the squares of two consecutive odd numbers is always a multiple of 8**

You could try it out:
$5^2 - 3^2 = 25 - 9 = 16 = 2 \times 8$

$17^2 - 15^2 = 289 - 225 = 64 = 8 \times 8$

You can try whatever consecutive numbers you like. It worked for the two examples here, but this does not prove it.

Even though examples don't prove a rule, they can help us to understand what the rule is saying.

Once you understand it, follow the rule using algebra.

Your solution should look something like this:

Let n be an integer (a whole number)

Then 2n would be an even number and (2n+1) would be an odd number

Consecutive odd numbers increase by 2 so (2n+3) would be the next odd number after (2n+1)

$(2n+3)^2 - (2n+1)^2 = (2n+3)(2n+3) - (2n+1)(2n+1)$

$= 4n^2 + 12n + 9 - (4n^2 + 4n + 1)$

$= 4n^2 + 12n + 9 - 4n^2 - 4n - 1$

$= 8n + 8$

$= 8(n+1)$ which is a multiple of 8 as required

Take care, it's easy to make mistakes with – signs or when multiplying out brackets

Try this one:

**Prove that the sum of the squares of two consecutive even numbers is always a multiple of 4**

Start by using some numbers to check if the rule works
Any suitable example
E.g. $4^2 + 6^2 = 16 + 36 = 52 = 4 \times 13$

Do you understand what the rule is saying?
You need to know what these words mean?
**Sum**
**Square**
**Consecutive**
**Multiple**

Let n be a whole number

How would you write an even number? $2n$

How would you write the next even number? $2n + 2$

Now square the two even numbers and add them together (be careful with brackets)

$(2n)^2 + (2n+2)^2$

Simplify your expression

$(2n)^2 + (2n+2)^2 = 4n^2 + (4n^2 + 8n + 4)$

$= 8n^2 + 8n + 4$

Now write your expression as two factors (one of the factors must be 4)

$= 4(2n^2 + 4n + 1)$ which is a multiple of 4 as required

Try this one by yourself:

**The mean of three consecutive numbers is always the middle number**

Try – (accept any suitable example) 3, 4, 5 mean = $(3 + 4 + 5) \div 3 = 12 \div 3 = 4$

Let n be an integer

The three consecutive numbers are n, $(n + 1)$ & $(n + 2)$ (other sets are possible)

Mean = $(n + n + 1 + n + 2) \div 3 = (3n + 3) \div 3 = n + 1$ this is the middle number as required

Now try this one:

**The product of two consecutive odd numbers is always an odd number**

Example: 3 x 5 = 35

Let n be an integer
Then (2n + 1) is an odd number

(2n + 1) and (2n + 3) are consecutive odd numbers

$(2n + 1) \times (2n + 3) = (2n + 1)(2n + 3)$

$= 2n^2 + 8n + 3$

$= 2(n^2 + 4n + 1) + 1$ which is an odd number as required

Here is a nice little number trick (It may be helpful on your non-calculator paper)

To square a number that ends .5 follow these steps:

1. Multiply the whole number part by the next whole number
2. Write this down
3. Now put .25 at the end of your number

Example
Work out $3.5^2$

3 x 4 = 12

So 12.25 is the answer

Try the trick yourself
Work out $6.5^2$

6 x 7 = 42

So 42.25 is the answer

Try one of your own
Work out ___$.5^2$

Check these answers on a calculator

Were they all right? Yes

Why does this work? We can answer this question with a proof:

Let n be a whole number

Then $n.5^2$ is correctly written as $(n + 0.5)^2$

$(n + 0.5)^2 = (n + 0.5)(n + 0.5) = n^2 + n + 0.25 = n(n+1) + 0.25$ as required

**Prime numbers**

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
| 53 | 59 | 61 | 67 | 71 | 73 | 79 | 83 | 89 | 97 | 101 | 103 | 107 | 109 | 113 |
| 127 | 131 | 137 | 139 | 149 | 151 | 157 | 163 | 167 | 173 | 179 | 181 | 191 | 193 | 197 |
| 199 | 211 | 223 | 227 | 229 | 233 | 239 | 241 | 251 | 257 | 263 | 268 | 271 | 277 | 281 |
| 283 | 293 | 307 | 311 | 313 | 317 | 331 | 337 | 347 | 349 | 353 | 359 | 367 | 373 | 379 |

| n | n^2 + n + 41 | Prime |
|---|---|---|
| 1 | 43 | ✓ |
| 2 | 47 | ✓ |
| 3 | 53 | ✓ |
| 4 | 61 | ✓ |
| 5 | 71 | ✓ |
| 6 | 83 | ✓ |
| 7 | 97 | ✓ |
| 8 | 113 | ✓ |
| 9 | 131 | ✓ |
| 10 | 151 | ✓ |
| 11 | 173 | ✓ |
| 12 | 197 | ✓ |
| 13 | 223 | ✓ |
| 14 | 251 | ✓ |
| 15 | 281 | ✓ |
| 16 | 313 | ✓ |
| 17 | 347 | ✓ |
| 18 | 383 | ✓ |
| 19 | 421 | ✓ |
| 20 | 461 | ✓ |
| 21 | 503 | ✓ |
| 22 | 547 | ✓ |
| 23 | 593 | ✓ |
| 24 | 641 | ✓ |
| 25 | 691 | ✓ |
| 26 | 743 | ✓ |
| 27 | 797 | ✓ |
| 28 | 853 | ✓ |
| 29 | 911 | ✓ |
| 30 | 971 | ✓ |

| n | n^2 + n + 41 | Prime |
|---|---|---|
| 30 | 971 | ✓ |
| 31 | 1033 | ✓ |
| 32 | 1097 | ✓ |
| 33 | 1163 | ✓ |
| 34 | 1231 | ✓ |
| 35 | 1301 | ✓ |
| 36 | 1373 | ✓ |
| 37 | 1447 | ✓ |
| 38 | 1523 | ✓ |
| 39 | 1601 | ✓ |
| 40 | 1681 | 1681 = 41x41 |
| 41 | 1763 | 1763 = 41x43 |
| 42 | 1847 | ✓ |
| 43 | 1933 | ✓ |
| 44 | 2021 | 2021 = 43x47 |
| 45 | 2111 | ✓ |
| 46 | 2203 | ✓ |
| 47 | 2297 | ✓ |
| 48 | 2393 | ✓ |
| 49 | 2491 | 2491=47x53 |
| 50 | 2591 | ✓ |
| 51 | 2693 | ✓ |
| 52 | 2797 | ✓ |
| 53 | 2903 | 2903=23x91 |
| 54 | 3011 | ✓ |
| 55 | 3121 | ✓ |
| 56 | 3233 | 3233=53x61 |
| 57 | 3347 | ✓ |
| 58 | 3463 | ✓ |
| 59 | 3581 | ✓ |