Advanced Mathematics Support Programme®

Where did Pythagoras' Theorem originate? Teacher Notes

Purpose of session:
The key emphasis of the activity is to show how the theorem we refer to as 'Pythagoras' Theorem' has a long and rich history which pre-dates Pythagoras. In highlighting this, students will appreciate how maths has developed in different cultures in different eras of history. Students will see the appearance of the theorem in ancient Babylonian and Chinese mathematics, work on some ancient problems, prove Pythagoras' Theorem, and generate Pythagorean Triples.

Audience:
This activity is designed to be used with a group of students who have met Pythagoras' Theorem and have begun to use it when solving unfamiliar problems.
'*…use Pythagoras' Theorem…in similar triangles to solve problems involving right-angled triangles*' (KS3 Programme of Study, KS4 Programme of Study).

Session Outline:
- Opening problem: An ancient problem from Susa taken from the Old Babylonian Period (Slides 3-4)
- Two further ancient problems, one from Mesopotamia and one from China (Slides 5-6)
- Chinese origins: The *Jui Zhang* and the *Gougu* diagram and proof (Slides 7- 11)
- Babylonian origins: Plimpton 322 tablet and Pythagorean Triples (Slides 12-16)

Resources:
- Slides for classroom display. These are reproduced below (pages 2-5) with notes for teachers.
- Student Handout. A simple two-page handout (pages 6-7 below) containing the key problems and images from the session with space for students to take notes and work on problems.

<table>
<tr><th>Slide (image and number)</th><th>Guidance notes for teachers</th></tr>
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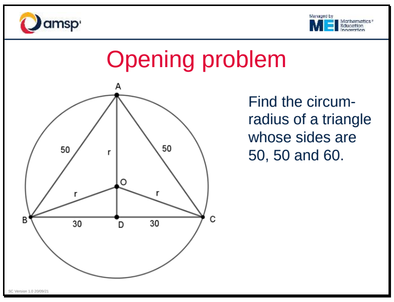
Slide 3



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Ask students how they tackled the problem – the expectation will be that they use Pythagoras' Theorem.

Note that a circum-radius of a cyclic polygon is the radius of the circle inside which the polygon can be inscribed.

<u>Possible solution:</u>

Applying Pythagoras' Theorem to ABD: $(AD)^2 + 30^2 = 50^2$, so $AD = 40$.

Applying Pythagoras' Theorem to OBD: $r^2 = (40 - r)^2 + 30^2$, so $80r = 2500 \Rightarrow r = \frac{125}{4} = 31.25$

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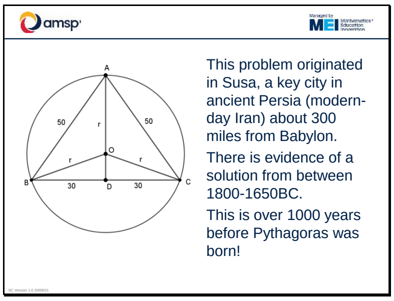
Slide 4



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Note that Susa was a key city which was located about 300 miles away from Babylon, the capital of ancient Persia. This is in modern-day Iran.

Pythagoras was born in around 570 BC.

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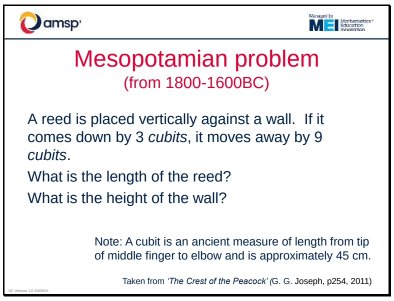
Slide 5



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The wording is potentially ambiguous so easiest to imagine the reed initially standing up alongside the wall with it's top higher than the wall. The reed is then lowered so its end rests on the top of the wall.

<u>Solution:</u>

Length of reed $= l$.

$(l - 3)^2 + 9^2 = l^2$ giving Reed (l) 15 cubits long. Wall is 12 cubits high.

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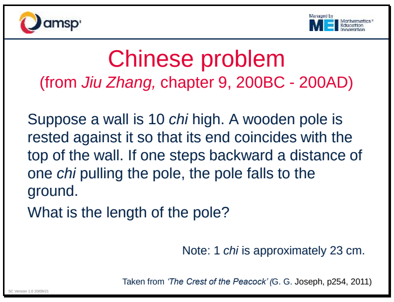
Slide 6



NB: Notes about The *Jiu Zhang* follow on next couple of slides.

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<u>Possible solution:</u> Let x be the distance the foot of the pole is from the wall initially. Then length of pole, $l = x + 1$.

Applying Pythagoras' Theorem gives: $(x + 1)^2 = x^2 + 10^2$, so $x = 49.5$. Hence, length of pole $l = 50.5$ *chi.*

Note the similarity of the two problems on Slides 5&6 which originate from different parts of the world.

Possible discussion point:
Why do you think there's such similarity? What might this mean?
This could lead to discussions about:

- how different cultures work on similar problems (the implication being that no culture has a monopoly on (mathematical) knowledge);
- development of maths knowledge (all build on work of others);
- transmission of maths (or any knowledge) in written, oral, etc form;
- travel around the ancient world (some scholars say that Pythagoras travelled widely and may have seen 'his' theorem in other parts of the world);
- etc…

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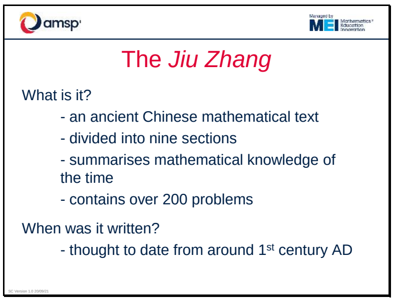
Slide 7



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The *Jiu Zhang* summarises the Chinese mathematical knowledge of the time and contains over 200 problems. It has nine sections, the final one of which is called *Gougu* meaning Base-Height.

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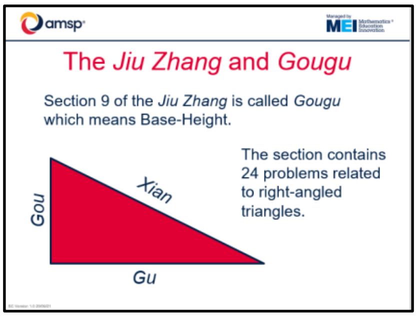
Slide 8



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Gou = shortest side of right-angled triangle

Gu = longer side

Xian = hypotenuse

Note that one of the 24 problems was the wall and wooden pole problem students just worked on (Slide 6).

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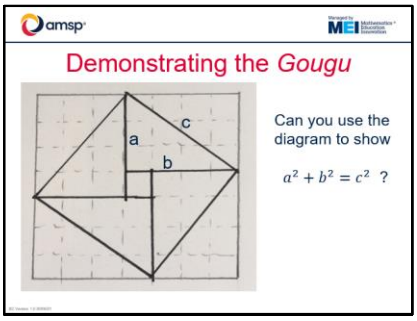
Slide 9



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A diagram similar to the image above appears in the *Zhou Bi*, an early (c. 1000BC) Chinese text on astronomy and maths.

Note: The “Gougu Rule” is referred to in the New Zealand curriculum alongside Pythagoras’ Theorem: *https://nzmaths.co.nz/resource/gougu-rule-or-pythagoras-theorem*

The diagram on the slide is produced on squared paper and so students could use a square-counting method to show it is true. There is an opportunity for students to reason in different ways. There are at least two different ways of thinking about this image algebraically to demonstrate the result. These are displayed on the next two slides.

The **Student Handout** (see pages 6-7) contains this image.

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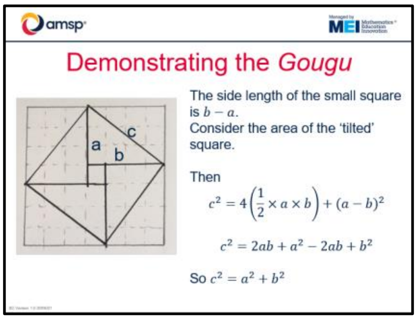
Slide 10



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Possible proof:

The length of the small square is $b - a$.

The ‘tilted’ square has side length c, and so area c^2.

But this area could also be viewed as the sum of the areas of four triangles, base a, height b, and the small square.

This gives: $c^2 = 4\left(\frac{1}{2} \times a \times b\right) + (b - a)^2$

So $c^2 = 2ab + b^2 - 2ab + a^2$

And $c^2 = a^2 + b^2$

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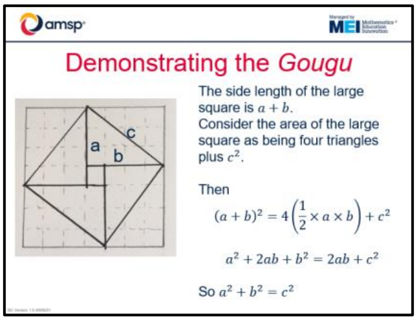
Slide 11



Possible proof:

The side length of the large square is $a + b$. So area is $(a + b)^2$.

This area could also be viewed as the sum of four triangles base a, height b and the 'tilted' square of side length c.

This gives: $(a + b)^2 = 4\left(\frac{1}{2} \times a \times b\right) + c^2$

So $a^2 + 2ab + b^2 = 2ab + c^2$

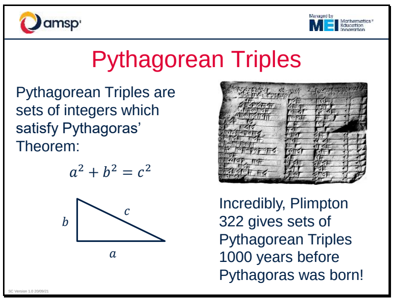
And $c^2 = a^2 + b^2$

Slide 12

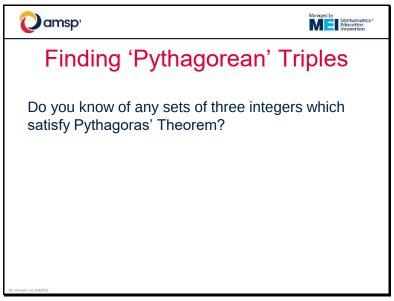
The tablet, housed in a collection at Columbia University, USA, dates from around 1800 BC and shows Pythagorean Triples with integer solutions.

Image taken from *https://www.alamy.com/stock-photo-plimpton-322-140400139.html*

Slide 13

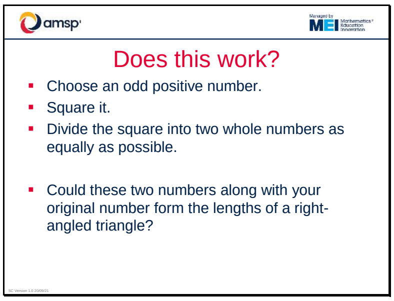


Slide 14



Opportunity for students to recall common Triples such as $3 - 4 - 5$, $6 - 8 - 10$ and $5 - 12 - 13$.

Slide 15



E.g. $13^2 = 169 = 84 + 85$. $13 - 84 - 85$ is a Pythagorean Triple.

This could be used as a class task with resulting Pythagorean Triples being collected on the board and checked by the class.

Extension: Why does this work? Can students prove it always works?

<u>Approach to algebraic proof:</u>

Odd number = $2n + 1$

Square this = $(2n+1)^2$

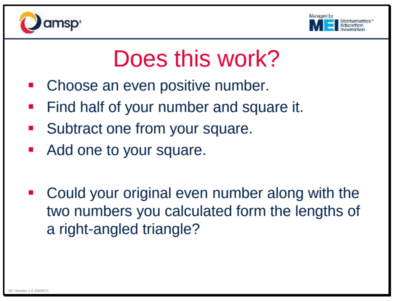
Two integers nearest to half of the square are $\frac{(2n+1)^2+1}{2}$ and $\frac{(2n+1)^2-1}{2}$

Applying Pythagoras' Theorem gives:

$$(2n+1)^2 + \left(\frac{(2n+1)^2-1}{2}\right)^2 = \frac{(2n+1)^4 + 2(2n+1)^2 + 1}{4}$$

$$= \left(\frac{(2n+1)^2+1}{2}\right)^2$$

Slide 16



E.g. $24 = 2 \times 12, 12^2 - 1 = 143, 12^2 + 1 = 145$. $24 - 143 - 145$ is a Pythagorean Triple.

This could be used as a class task with resulting Pythagorean Triples being collected on the board and checked by the class.

Extension: Why does this work? Can students prove it always works?

<u>Approach to algebraic proof:</u>

Let your even number be $2n$.

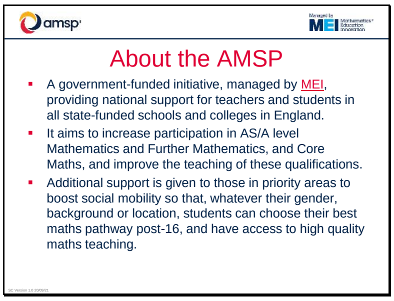
Half of $2n$ is n, and the square of this is n^2.

Subtracting one gives $n^2 - 1$, and adding one gives $n^2 + 1$.

Applying Pythagoras' Theorem gives:

$$(2n)^2 + (n^2-1)^2 = 4n^2 + n^4 - 2n^2 + 1 = n^4 + 2n^2 + 1 = (n+1)^2$$

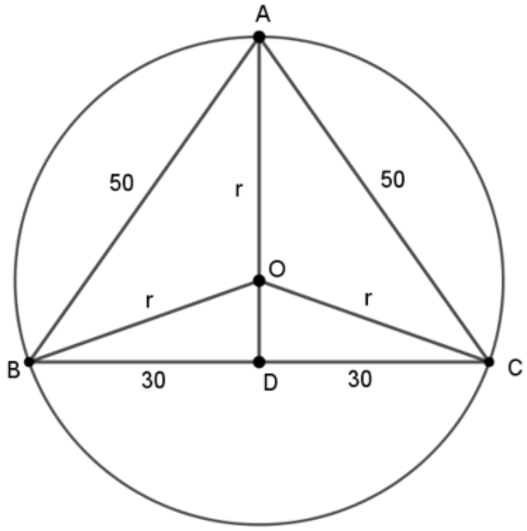
Slides 17 & 18



Information & contact details about the AMSP.

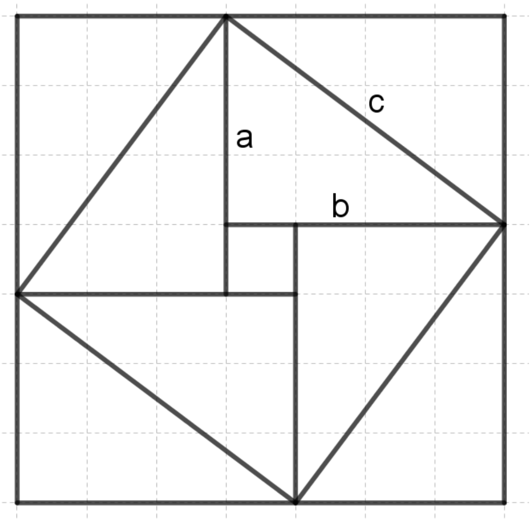
Where did Pythagoras' Theorem originate?

Find the circum-radius of a triangle whose sides are 50, 50 and 60.



An ancient Mesopotamian problem (1800-1600BC)	An ancient Chinese problem (200BC – 200AD)
A reed is placed vertically against a wall. If it comes down by 3 *cubits* [so the end of the reed now rests on the top of the wall], it moves away by 9 *cubits*. What is the length of the reed? What is the height of the wall?	Suppose a wall is 10 *chi* high. A wooden pole is rested against it so that its end coincides with the top of the wall. If one steps backward a distance of one *chi* pulling the pole, the pole falls to the ground. What is the length of the pole?

Demonstrating the Gougu Theorem



Generating Pythagorean Triples