Advanced Mathematics Support Programme®

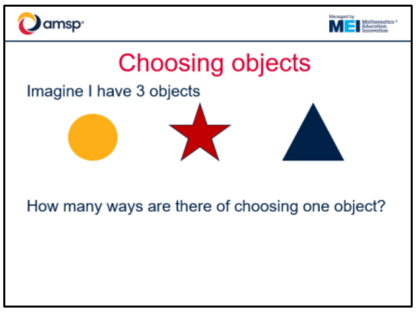
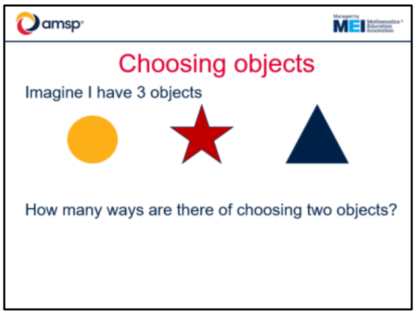
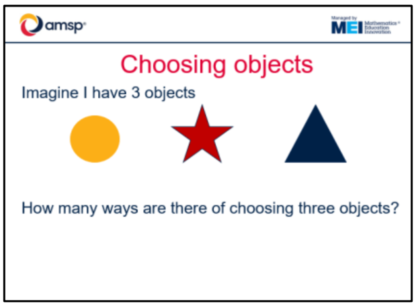
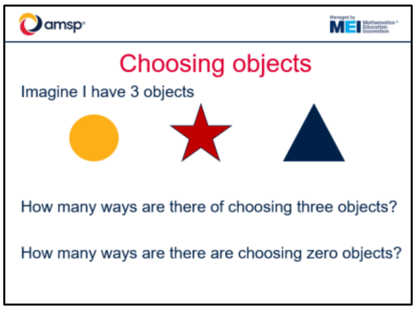
Whose triangle is it? Teacher Notes

Purpose of session:
The key emphasis of the activity is to demonstrate the many interesting patterns involved in the array of numbers we usually refer to as 'Pascal's triangle'. The activity will also explore some of the historical origins of this triangular array.

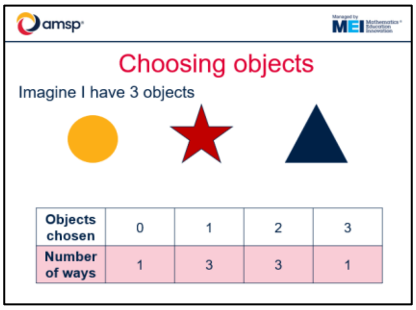
Audience:
All activities can be worked on and discussed numerically and would be suitable for students with limited, if any, prior knowledge of Pascal's triangle. Some knowledge of different categories of number (e.g. triangular numbers, tetrahedral numbers) would be useful. For the optional final activities, the ability to expand brackets or produce a tree diagram are required.

Session Outline:
- Choosing objects (Slides 3-9) – students look at how many ways there are to choose a certain number of objects from the list of objects available. This will develop an introduction to Pascal's triangle.
- The history of "Pascal's triangle" (Slides 10 - 11)
- What next? (Slides 12-13) – two possible extension activities to show other ways in which Pascal's triangle occurs in areas of the maths curriculum

Resources:
- Slides for classroom display. These are reproduced below (pages 2 - 6) with notes for teachers.
- Student Handout. A simple three-page handout (pages 7 - 9 below) containing the key problems and images from the session with space for students to take notes and work on problems.

Slide (image and number)	Guidance notes for teachers
Slide 3 	This will seem quite obvious – students may wonder why you are asking this question, but it's worth noting that this is taking us somewhere! The answer is 3. **For slides 3 – 8 copies of the shapes are available on the Student Handout on pages 7 - 8 below.**
Slide 4 	Here a bit more strategy is needed – students could list them i.e., circle-star; circle-triangle; star-triangle so there are again 3 ways. It is important students understand that when choosing objects the order of choosing is not important (e.g., circle-star and star-circle are the same) and that objects are not replaced once they are chosen (e.g. you could not get circle-circle or star-star).
Slide 5 	Choosing 3 objects from 3 available objects obviously just has one possible way. This is a very basic example of a branch of maths called combinatorics, which is focused on methods of counting in order to solve problems.
Slide 6 	It might be worth mentioning now that we could also consider how many ways we could choose zero objects from the list – this is slightly abstract but essentially there is always 1 way of choosing no objects – you just don't choose any!

Slide 7



Here are the results for choosing 0, 1, 2 and 3 objects from 3 objects.

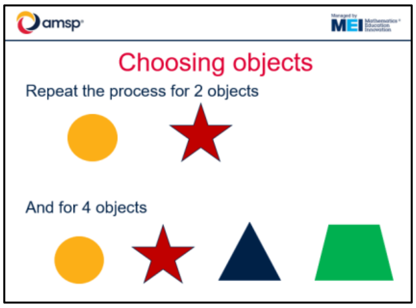
Students might have spotted some symmetry here and might be able to explain why choosing 1 object and choosing 2 objects has the same number of possibilities.

For example, in the table below the first column might represent the object that is chosen or the object that is *not* chosen.

Circle	Star, Triangle
Star	Circle, Triangle
Triangle	Circle, Star

In either case there are three possibilities, the first column gives the options for choosing one object and the second column gives the options for choosing two objects. We can see that the possibilities naturally 'pair up' and so there are 3 options in each case.

Slide 8



Ask the students to think about how many ways they could choose:

- zero, one or two objects from 2 objects (answer: 1, 2, 1)
- zero, one, two, three or four objects from 4 objects (answer: 1, 4, 6, 4,1)

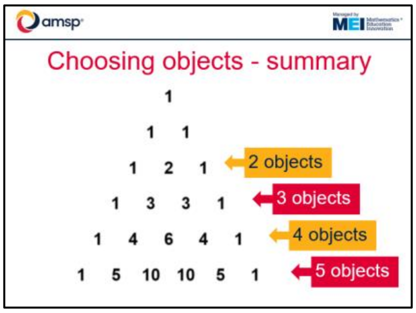
Encourage students to produce a table of results for each of these scenarios, similar to that on the last slide.

What would happen with just one object to choose from? (answer: 1,1)

What if there were 5 objects? Students might try this and think about how to list the outcomes systematically. They might use the numbers 1 to 5 instead of drawing shapes when listing the combinations.

What happens in the table of results each time? Are there any patterns within each table?

Slide 9



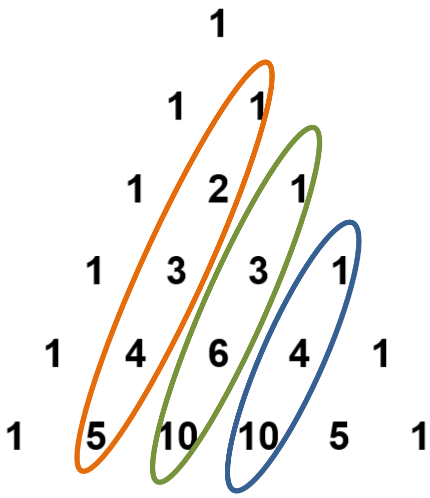
Encourage students to see how the individual rows of the triangular array are copies of the tables for each of their number of objects considered earlier.

What might the first two rows represent? The first '1' is a bit abstract representing the number of ways to choose 0 objects from 0 objects! Perhaps not something to focus on in depth but interesting to note the patterns that emerge **between** the rows.

- Could they predict the next row without actually working out individually all of the different combinations? As the number of objects increases, why might studying the patterns be a better approach? **See Student Handout on page 8 below (note it is worth having this printed separately to pages 6 – 7 of the Student Handout so that students don't look ahead and see the patterns).**
- Based on the work earlier, can students explain why the triangle is symmetrical e.g. how are the two 4's on the line 1,4,6,4,1 related to each other? The table in the Teacher Notes for slide 7

above might be a useful approach here. Students could explain why the number of possibilities for choosing one object and choosing three objects out of four objects is the same.

- Do students spot any other patterns within the triangle? Encourage them to look diagonally and/or calculate the sum of the numbers in each row.



For example, the diagonal highlighted in orange is the natural numbers or counting numbers. The green diagonal shows the triangular numbers, and the blue diagonal shows the tetrahedral numbers (see *this resource* for how the tetrahedral numbers are formed and how they are linked to triangular numbers).

The individual row totals are: 1, 2, 4, 8, 16, 32, … which are powers of 2, the power being the row number, assuming the triangle starts with “row zero”.

There are more patterns to explore on Pascals’ triangle - *this resource* might be a useful source of extension activities.

Slide 10

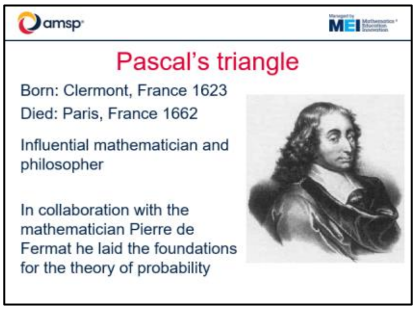


Image & info taken from *https://mathshistory.st-andrews.ac.uk/Biographies/Pascal/*

The triangle on the previous slide is commonly referred to as Pascal’s triangle…. though that may not be crediting this work fairly, as we’ll see on the next slide!

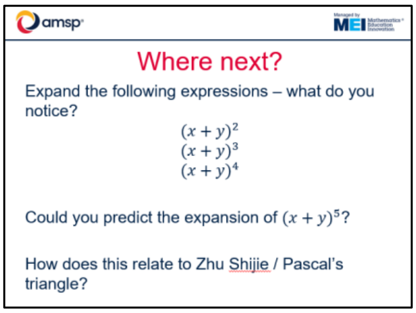
Slide 11

We know little about Zhu Shijie (also known as Chu Shih-Chieh)—not precisely even when he was born or when he died. He lived near modern Beijing, but he seems to have spent some twenty years as a wandering scholar who earned his living by teaching mathematics. (see https://www.worldsmostinfluentialpeople.com/zhu-shijie). In 1303 he published a book called 'Precious Mirror of the 4 elements' in which he included a diagram which later, in the west, became known as Pascals' triangle (published c1654 – 1655) – see copy on this slide. Zhu's book entitles it, "Diagram of the Old Method for finding Eighth and Lower Powers," suggesting that it was known at least for some time before he published it.…

More info about this publication here *https://mathshistory.st-andrews.ac.uk/Biographies/Zhu_Shijie/*

This history suggests that the triangle we know as "Pascal's" triangle was known about possibly prior to 1000 AD – several hundred years prior to Pascal publishing it.

Slide 12

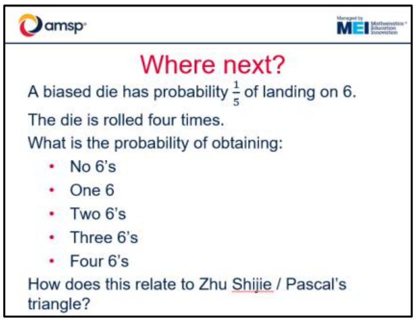


This is one of two optional extension activities that you might use to show the range of ways in which Zhu Shijie / Pascal's triangle is used within the curriculum.

The coefficients of each of the expressions are represented in the row of the triangle which correspond to the power (taking the first row, with a single 1, as "row zero" i.e. $(x + y)^0$)

More extension: Students might extend this work further by researching factorial notation and the use of the nCr button on their calculator.

Slide 13



This is one of two optional extension activities that you might use to show the range of ways in which Zhu Shijie / Pascal's triangle is used within the curriculum.

Students could use a tree diagram to show the different options. The number of ways to obtain each of the outcomes listed corresponds to a row of Pascal's triangle (row 4, since there are 4 rolls of the dice, assuming the row with a single '1' is "row zero").

For example, the number of ways to get two 6's is shown in the table – there are 6 ways.

First roll	Second roll	Third roll	Fourth roll
6	6	Not 6	Not 6
6	Not 6	6	Not 6
6	Not 6	Not 6	6
Not 6	6	6	Not 6
Not 6	6	Not 6	6
Not 6	Not 6	6	6

<table>
<tr>
<td></td>
<td>So the <u>probability</u> of obtaining two 6’s is $6 \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^2 = 0.1536$ i.e. there are six ways of rolling two sixes (each with probability $\frac{1}{5}$) and two non-sixes (each with probability $\frac{4}{5}$).</td>
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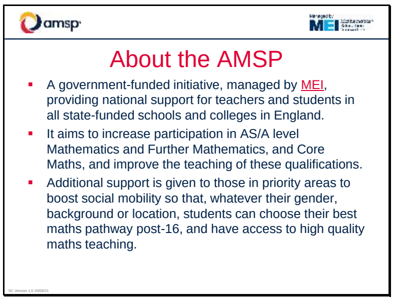
<td>Slides 14 & 15



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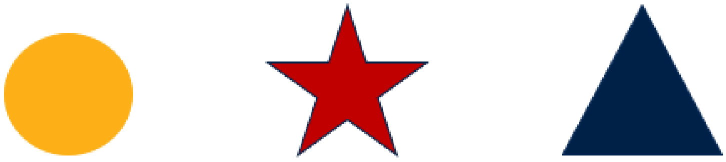
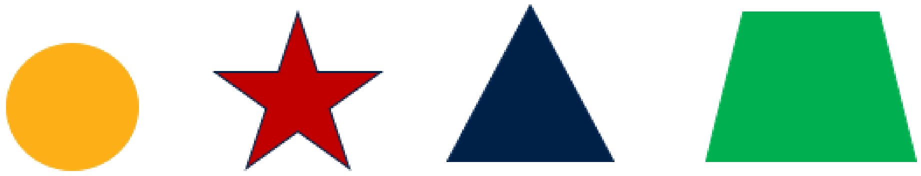
Whose triangle is it?

Choosing from 3 objects



Choosing from 2 objects

Choosing from 4 objects



Pascal's triangle