Advanced Mathematics Support Programme®

Problem Solving in Further Maths

Phil Chaffé

Chris Saker

Desmos activities at

student.desmos.com
Code: DCP FUY

Continuing Professional Development Standard

National Centre for Excellence in the Teaching of Mathematics

3.1.2 OT2: Mathematical problem solving

	Content
OT2.1	Recognise the underlying mathematical structure in a situation and simplify and abstract appropriately to enable problems to be solved.
OT2.2	Construct extended arguments to solve problems presented in an unstructured form, including problems in context.
OT2.3	Interpret and communicate solutions in the context of the original problem.
OT2.6	Understand the concept of a mathematical problem solving cycle, including specifying the problem, collecting information, processing and representing information and interpreting results, which may identify the need to repeat the cycle.
OT2.7	Understand, interpret and extract information from diagrams and construct mathematical diagrams to solve problems, including in mechanics.

- AO3: Solve problems within mathematics and in other contexts. Students should be able to:
  - translate problems in mathematical and non-mathematical contexts into mathematical processes
  - interpret solutions to problems in their original context, and, where appropriate, evaluate their accuracy and limitations
  - translate situations in context into mathematical models
  - use mathematical models
  - evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them.
- Where questions/tasks targeting this assessment objective will also credit students for the ability to 'use and apply standard techniques' (AO1) and/or to 'reason, interpret and communicate mathematically' (AO2) an appropriate proportion of the marks for the question/task must be attributed to the corresponding assessment objective(s).

AQA AS Level Further Mathematics

Assessment objectives (AOs)	Component weightings (approx %)		Overall weighting (approx %)
	Paper 1	Paper 2	
AO3	15	25	20

AQA A Level Further Mathematics

Assessment objectives (AOs)	Component weightings (approx %)			Overall weighting (approx %)
	Paper 1	Paper 2	Paper 3	
AO3	20	20	35	25

Where is the problem solving in that?

You are going to see 4 Further Maths Questions from the Specimen papers

For each one try to identify where the problem-solving element of the question is.

student.desmos.com code: DCP FUY

We are now working individually at

student.desmos.com

Activity code: DCP FUY

Presenter Warning!

The answers start on the next slide

OCR (MEI) AS

The cubic equation $x^3 - 4x^2 + px + q = 0$ has roots α, $\frac{2}{\alpha}$ and $\alpha + \frac{2}{\alpha}$.

Find

- the values of the roots of the equation,
- the value of p.

[7]

AO1	AO2	AO3(PS)	AO3(M)	Total
4	0	3	0	7

Use and apply standard techniques

Solve problems within mathematics and in other contexts

AQA A

Evaluate the improper integral $\int_0^{\infty} \frac{4x - 30}{(x^2 + 5)(3x + 2)} \mathrm{d}x$, showing the limiting process used.

Give your answer as a single term.

[8 marks]

Splits integrand into partial fractions of the form $\frac{Ax + B}{x^2 + 5} + \frac{C}{3x + 2}$	AO3.1a	M1

Edexcel AS

An octopus is able to catch any fish that swim within a distance of 2 m from the octopus's position.

A fish F swims from a point A to a point B.

The octopus is modelled as a fixed particle at the origin O.

Fish F is modelled as a particle moving in a straight line from A to B.

Relative to O, the coordinates of A are $(-3, 1, -7)$ and the coordinates of B are $(9, 4, 11)$, where the unit of distance is metres.

(a) Use the model to determine whether or not the octopus is able to catch fish F.

(7)

(b) Criticise the model in relation to fish F.

(1)

(c) Criticise the model in relation to the octopus.

(1)

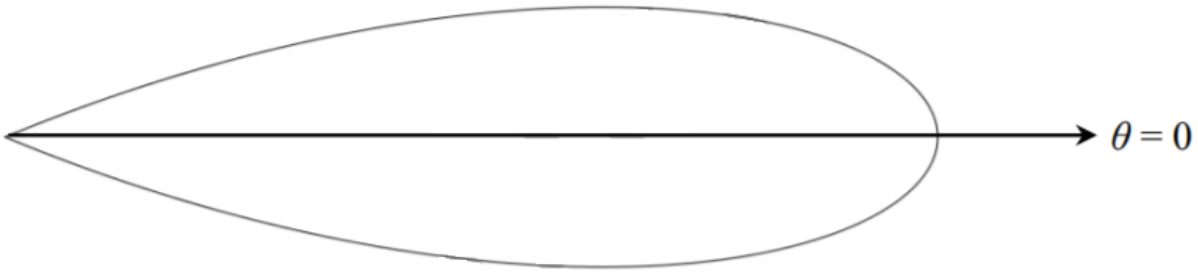
$\overrightarrow{AB} = \begin{pmatrix} 9 \\ 4 \\ 11 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \left\{ = \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} \right\}$ or $\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$	M1	3.1a
$\left\{ \overrightarrow{OF} = \mathbf{r} = \right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$	M1	1.1b
$\left\{ \overrightarrow{OF} \bullet \overrightarrow{AB} = 0 \Rightarrow \right\} \begin{pmatrix} -3+12\lambda \\ 1+3\lambda \\ -7+18\lambda \end{pmatrix} \bullet \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} = 0$ $\Rightarrow -36+144\lambda+3+9\lambda-126+324\lambda=0 \Rightarrow 477\lambda-159=0$	dM1	1.1b
$\Rightarrow \lambda = \frac{1}{3}$	A1	1.1b

$\left\{\overrightarrow{OF} =\right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and minimum distance $= \sqrt{(1)^2 + (2)^2 + (-1)^2}$	dM1	3.1a
$= \sqrt{6}$ **or** 2.449…	A1	1.1b
> 2, so the octopus is not able to catch the fish F	A1ft	3.2a
e.g. Fish F may not swim in an exact straight line from A to B Fish F may hit an obstacle whilst swimming from A to B Fish F may deviate his path to avoid being caught by the octopus	B1	3.5b
	(1)	
e.g. Octopus is effectively modelled as a particle – so we may need to look at where the octopus's mass is distributed Octopus may during the fish F's motion move away from its fixed location at O	B1	3.5b

OCR A

In this question you must show detailed reasoning.

The diagram below shows the curve $r = 2\cos 4\theta$ for $-k\pi \leq \theta \leq k\pi$ where k is a constant to be determined.



Calculate the exact area enclosed by the curve. **[6]**

DR

$r = 0 \Rightarrow k = \frac{1}{8}$

$A = \frac{1}{2}\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} r^2 \, d\theta = \frac{1}{2}\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} (2\cos 4\theta)^2 \, d\theta$

$= 2\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} \cos^2 4\theta \, d\theta$

$= \int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi} (1 + \cos 8\theta) \, d\theta = \left[\theta + \frac{\sin 8\theta}{8}\right]_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi}$

$= \left(\frac{1}{8}\pi + \frac{\sin \pi}{8}\right) - \left(-\frac{1}{8}\pi + \frac{\sin(-\pi)}{8}\right) = \frac{1}{4}\pi$

B1	2.2a		
M1	3.1a	Limits not required	Must be seen
A1	1.1	Correct form to be integrated	
A1	1.1	Correct indefinite integral	
M1	3.1a	Using correct limits	Must be seen
A1	1.1	Must show $f(\frac{1}{8}\pi) - f(-\frac{1}{8}\pi)$	**A0** for decimal answer
[6]			

Question	AO1	AO2	AO3(PS)	AO3(M)	Total
3	3	1	2	0	6

Some important questions

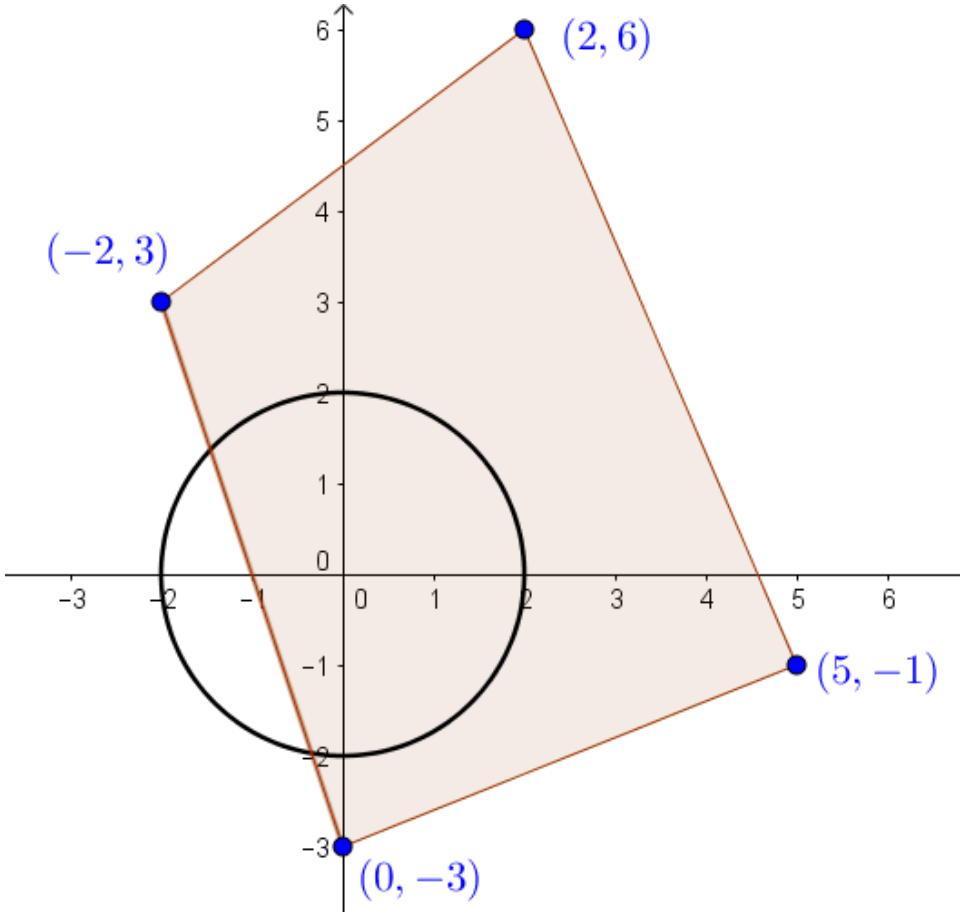
- Do we need to worry about the problem solving content in AS and A level Further Mathematics?
- Is there any point in teaching problem solving skills for the new AS and A level?
- How do you teach students so that they develop problem solving skills?

Squashed

How many times must the transformation given by the matrix

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

be applied to the quadrilateral so that the resultant quadrilateral is inside the circle shown?



The four stages

1. Being able to ask yourself questions is key to being able to start the problem solving process.
    a. Gives students ‘permission’ to do this.
    b. Provides help (they are just starting out).
    c. Provides focus.
2. Thinking about all of the skills that could be used helps students to be critical about methods and how useful/efficient they are.
    a. Reminds students of relevant skills
    b. Starts to give focus on the calculations that may be done

The four stages

3. Suggesting ideas for solving the problem helps the students develop ‘forward thinking’ skills and critically assess possible paths
    a. Develops ‘hazy forward thinking’
    b. Allows for methods to go nowhere and be abandoned
    c. Allows the effectiveness of different methods to be discussed
4. This is the gateway to solving the problem.
    a. Students have to identify what to actually do….
    b. …. and then do it!

We are now working individually at

student.desmos.com

Activity code: DCP FUY

Why have a structure?

- It allows students to learn what is expected when problem solving
  - Learning to play the game
- It allows for reflection on the appropriateness of various methods
  - What does this tell me?
  - How will it help?
- It allows for students to learn from their peers
- It provides a framework that students can learn to apply on their own. (With repetition)

How could this problem be used?

- As an introduction to enlargement/reduction by linear transformations
  - Repeated transformations (matrix multiplication)
    - What happened?
    - How can we do this more efficiently?
- As an introduction to the determinant
  - Repeated transformations (matrix multiplication)
  - ‘Discovering’ the determinant’s use as a scale factor

How could this problem be used?

- As a ‘use of technology’ activity
  - Using Geogebra
  - Observing what happens with each transformation
- As an assessment of skills learned during the ‘linear transformations’ lessons
  - Identifying uses of the techniques covered
  - Linking concepts and skills

Problem-solving shorts

- There are 9 problem solving shorts
- Choose 1 or two of them
  - You may not like some of them – don't worry
- For each one think
  - How would you use this?
  - How would you introduce it?
  - What would you hope the students get out of it?
  - How could you extend the problem?

We are now working individually at

student.desmos.com

Activity code: DCP FUY

Identity crisis

- Calculate

$$\begin{pmatrix} \cos 73° & -\sin 73° \\ \sin 73° & \cos 73° \end{pmatrix} \begin{pmatrix} \cos 17° \\ \sin 17° \end{pmatrix}$$

Immovable

$(a, 3)$ is an invariant point of the transformation given by this matrix

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

What is the value of a?

Not moved

- $y = 2x$ is a line of invariant points of the transformation given by this matrix

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$

- What are the values of a and b?

Singular

- These matrices all have determinant 0.

$$\begin{pmatrix} a & b \\ 3 & 6 \end{pmatrix} \qquad \begin{pmatrix} 4 & a \\ b & 8 \end{pmatrix} \qquad \begin{pmatrix} a & c \\ c & 1 \end{pmatrix}$$

- What is the value of a?

Scalers

- The transformations represented by these two matrices both have the same area scale factor of enlargement. What are the possible values of a?

$$\begin{pmatrix} a & 3 \\ 4 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix}$$

Esrevni

If

$$\begin{pmatrix} a+1 & 1 \\ 2a+1 & 2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}=\begin{pmatrix} e \\ f \end{pmatrix}$$

then what is

$$\begin{pmatrix} 2 & -1 \\ -2a-1 & a+1 \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} \quad ?$$

Last in, first out

$\mathbf{M}$ and $\mathbf{N}$ are square matrices with inverses $\mathbf{M}^{-1}$ and $\mathbf{N}^{-1}$ respectively.

In terms of $\mathbf{M}^{-1}$ and $\mathbf{N}^{-1}$ what is the inverse matrix of

$$\mathbf{MMMNNNNMN}$$

Move along

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 23 \end{pmatrix}$$

Use this to find a solution to the simultaneous equations

$$x + 2y = 9$$
$$3x + 4y = 23$$

Flipper

a, b, c, d are constants and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \end{pmatrix}$.

Find a solution of the following simultaneous equations

$$bx + ay = 12$$
$$dx + cy = 3$$

Sources of problems from AMSP and MEI

- All of the problems come from the Integral resources
  - integralmaths.org.uk

- Other problems can be found on the AMSP website
  - amsp.org.uk

About the AMSP

- A government-funded initiative, managed by MEI, providing national support for teachers and students in all state-funded schools and colleges in England.
- It aims to increase participation in AS/A level Mathematics and Further Mathematics, and Core Maths, and improve the teaching of these qualifications.
- Additional support is given to those in priority areas to boost social mobility so that, whatever their gender, background or location, students can choose their best maths pathway post-16, and have access to high quality maths teaching.

Contact the AMSP

01225 716 492

admin@amsp.org.uk

amsp.org.uk

Advanced_Maths