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Where do students go wrong with Proof by Induction?

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Continuing Professional Development Standard

National Centre for Excellence in the Teaching of Mathematics

A proof by induction

Prove for $n \in \mathbb{N}$ that

$$\sum_{r=1}^{n} 2r + 3 = n(n+4)$$

STEP 1

Base case: $n = 1$

LHS $= 2 \times 1 + 3 = 5$

RHS $= 1(1+4) = 5$

LHS=RHS $\therefore$ true for $n = 1$

STEP 2

Assume true for $n = k$

So $$\sum_{r=1}^{k} 2r + 3 = k(k+4)$$

STEP 3

Look at $n = k + 1$

$$\sum_{r=1}^{k+1} 2r + 3 = \left(\sum_{r=1}^{k} 2r + 3\right) + 2(k+1) + 3$$

$$= \quad k(k+4) \quad + 2k + 5$$

$$= k^2 + 6k + 5$$

$$= (k+1)(k+5)$$

$$= (k+1)\big((k+1)+4\big)$$

STEP 4

Hence, if the result is true for $n = k$, then it is also true for $n = k + 1$. Therefore, since it is true for $n = 1$, it is true for all integers $n \geq 1$.

Aims

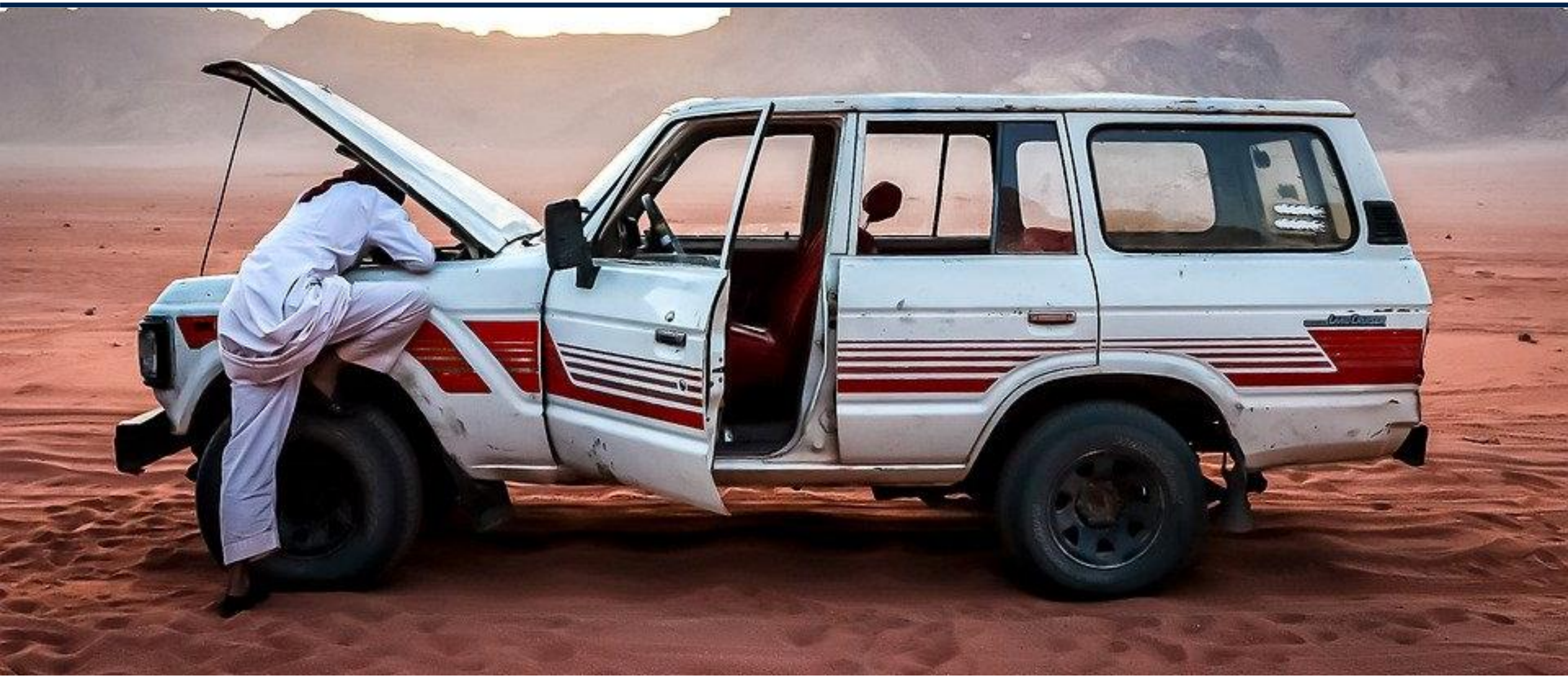
- To look at where students typically go wrong in producing a full proof by induction
- To share some thoughts on why they go wrong
- To suggest some strategies to address this, with a focus on proofs of divisibility

What the examiners say

- Mathematical induction is a well-known topic …
- The vast majority of students clearly knew how to structure a proof by induction …
- Most students showed a reasonably clear understanding of the steps required …

What the examiners **also** say

- … confounds many students
- … only a few were able to complete it
- … those few (under 15%) who gained full marks



Proof by induction

Where did it go wrong?

What's under the bonnet?

- Conceptual issues
- Technical issues
- 'Emotional' issues

An **incorrect** conclusion

“since it is true for $n = 1, n = k$ and $n = k + 1\ldots$”

Conceptual issues

- Conceptual faults
  - why do we just say it is true for $n = k$?
  - isn't that cheating?
  - what do the steps actually mean?

Making things hard

Prove for $n \in \mathbb{N}$ that $\sum_{r=1}^{n} r^2 + r = \frac{1}{3}n(n+1)(n+2)$

Inductive step:

$$\sum_{r=1}^{k+1} r^2 + r = \left(\sum_{r=1}^{k} r^2 + r\right) + (k+1)^2 + (k+1)$$

$$= \frac{1}{3}k(k+1)(k+2) + (k+1)^2 + (k+1)$$

$$= \frac{1}{3}k^3 + 2k^2 + \frac{11}{3}k + 2$$

$$= \ ?$$

Making things hard

Prove for $n \in \mathbb{N}$ that $\sum_{r=1}^{n} r^2 + r = \frac{1}{3}n(n+1)(n+2)$

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$$= \frac{1}{3}k(k+1)(k+2) + (k+1)^2 + (k+1)$$

$$\require{cancel}\cancel{= \frac{1}{3}k^3 + 2k^2 + \frac{11}{3}k + 2}$$

Factorise

$$= \frac{1}{3}(k+1)\{k(k+2) + 3(k+1) + 3\}$$

$$= \frac{1}{3}(k+1)(k^2 + 5k + 6)$$

$$= \frac{1}{3}(k+1)(k+2)(k+3)$$

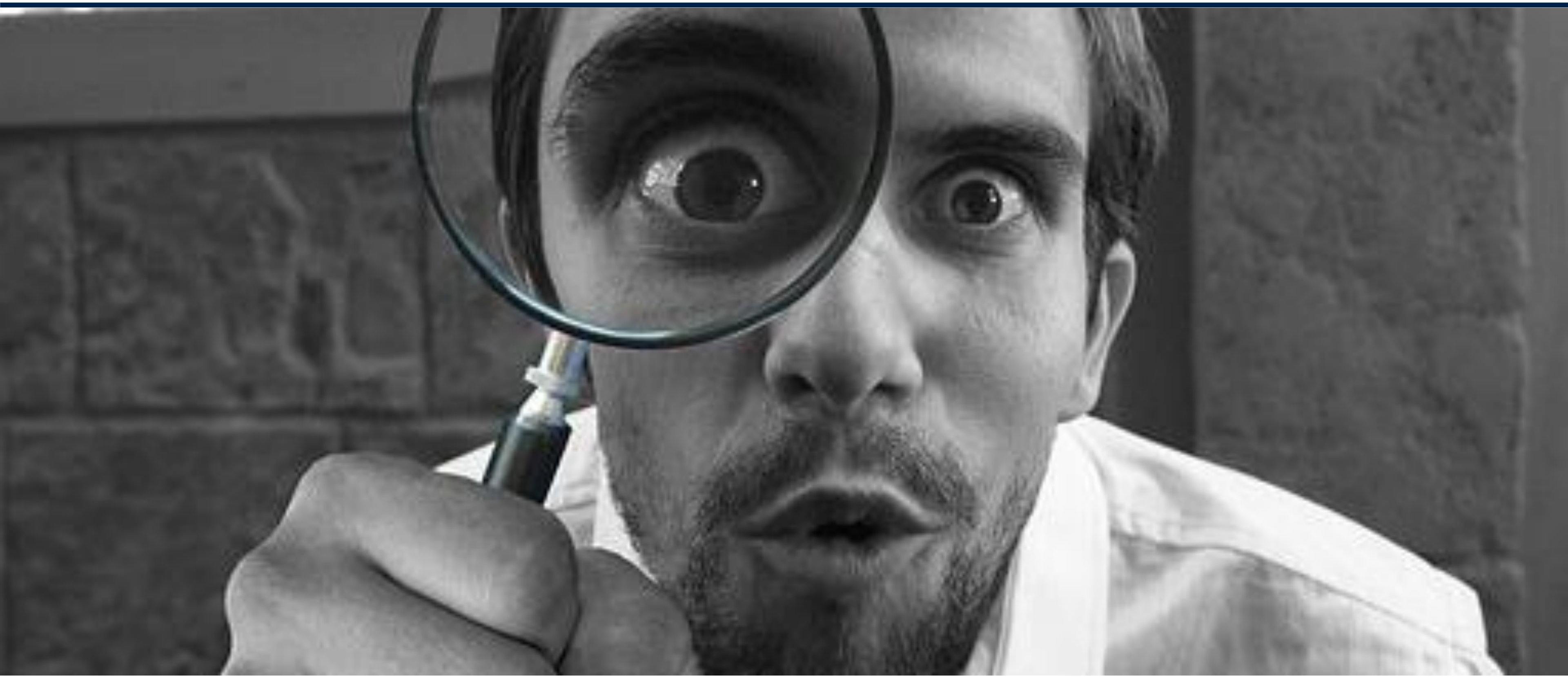
Technical issues

- Technical issues
  - fluency
  - factorise or expand
  - use (abuse?) of indices

Does everybody get a mark for the base case?

‘Emotional’ issues

- Mistrust – if you have to use words to explain then your algebra isn’t good enough!



Proof by induction

Focus on divisibility

Checking they have the skills…

- Number theory is curiously marginalised in the school mathematics curriculum

Technical issues

Do you know a divisibility test for…

- 2
- 3
- 4
- 5
- 6
- 7
- 8
- 9
- 10
- 11

What does it mean to say a number is divisible by 9?

Can you say the same thing a different way?

Have your students seen…

- a **proof** of the divisibility by 9 test?
- a **proof** that all prime numbers are either one more or one less than a multiple of 6 (or other similar result)?

How do you know this expression is divisible by 9?

$$9ab + 27b^2 - 6(3b + 6)$$

$$= 9(ab + 3b^2 - 2b - 4)$$

which is divisible by 9 if and only if the expression in the bracket is an integer.

Conceptual issues

Divisibility by 9

- A number *n* is divisible by 9 if the sum of its digits is divisible by 9

Proof for 3-digit numbers

Let the digits of n be a, b and c.

Then $$\begin{aligned} n &= 100a + 10b + c \\ &= 99a + a + 9b + b + c \\ &= (99a + 9b) + a + b + c \\ &= 9(11a + b) + a + b + c \\ &= 9\left(11a + b + \frac{a+b+c}{9}\right) \end{aligned}$$

which is a multiple of 9 if and only if $\frac{a+b+c}{9}$ is an integer, i.e. $a + b + c$ is divisible by 9.

Challenge: adapt this to prove a similar result for divisibility by 3.

All prime numbers are either one more or one less than a multiple of 6

Proof:

Every number can be written in one of the following ways, where k is an integer:

$6k-2$
$6k-1$
$6k$
$6k+1$
$6k+2$
$6k+3$
$6k+4$
$6k+5$

Conceptual issues

All prime numbers are either one more or one less than a multiple of 6

Proof:

Every number can be written in one of the following ways, where k is an integer:

$6k-2$

$6k-1$

$6k$

$6k+1$

$6k+2$

$6k+3$

Is this completely correct?

So numbers of the form $6k-2$, $6k$, $6k+2$ and $6k+3$ are not prime.

All prime numbers **greater than 3** are either one more or one less than a multiple of 6

Corrected!

Proof:

Every number can be written in one of the following ways, where k is an integer:

$6k-2=2(3k-1)$ which is a multiple of 2

$6k-1$

$6k$ which is a multiple of 6

$6k+1$

$6k+2=2(3k+1)$ which is a multiple of 2

$6k+3=3(2k+1)$ which is a multiple of 3

So numbers of the form $6k-2$, $6k$, $6k+2$ and $6k+3$ are not prime

All prime numbers **greater than 3** are either one more or one less than a multiple of 6

Proof:

Every number can be written in one of the following ways, where k is an integer:

$6k-2=2(3k-1)$ which is a multiple of 2

$6k-1$

$6k$ which is a multiple of 6

$6k+1$

$6k+2=2(3k+1)$ which is a multiple of 2

$6k+3=3(2k+1)$ which is a multiple of 3

So numbers of the form $6k-2$, $6k$, $6k+2$ and $6k+3$ are not prime (except 2 and 3).

Therefore the prime numbers greater than 3 must all be of the form $6k-1$ or $6k+1$, so that they are either one more or one less than a multiple of 6.

Can your students write down…

- a general even number?
- a general odd number?
- a multiple of 3?
- a multiple of k?
- 2 general consecutive numbers?
- 3 general consecutive numbers?

Technical issues

Try these…

- The product of two consecutive integers is even.
- The product of n consecutive integers is divisible by n if n is odd.
- For all $n \in \mathbb{N}$, $n^3 - n$ is a multiple of 3.
- If m is divisible by 2 and n is divisible by 3, then mn is divisible by 6.

Try these…

The product of two consecutive integers is even.

Proof:

Let n and $n+1$ be two general consecutive integers.

Then either n or $n+1$ is even.

So their product is even.

Try these…

The product of n consecutive integers is divisible by n if n is odd.

Proof:

Unsure where to start? Let's see what happens with $n = 3$…

Let $k - 1$, k and $k + 1$ be 3 general consecutive integers.

Then one of these numbers is a multiple of 3.

So their product is a multiple of 3.

Try these…

The product of n consecutive integers is divisible by n if n is odd.

Proof:

In general, let k, $k+1$, $k+2$, … $k+(n-1)$ be n general consecutive numbers.

Then one of these numbers is a multiple of n.

So their product is a multiple of n.

Try these…

For all $n \in \mathbb{N}$, $n^3 - n$ is a multiple of 3.

Proof:

$$n^3 - n = n(n^2 - 1)$$
$$= n(n+1)(n-1)$$

which is a product of 3 consecutive digits. One of these digits is a multiple of 3, so their product is a multiple of 3.

Try these…

If m is divisible by 2 and n is divisible by 3, then mn is divisible by 6.

Proof:

m is divisible by 2, so that $m = 2a$ for some $a \in \mathbb{Z}$.

n is divisible by 3, so that $n = 3b$ for some $b \in \mathbb{Z}$.

$\therefore mn = 2a \times 3b = 6ab$ which is divisible by 6.

Now let's try some Proof by Induction…

9 Prove by induction that $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers n. **[7]**

MEI A level FM Core Pure 2018

Now let's try some Proof by Induction…

9 Prove by induction that $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers n. **[7]**

Let $u_n = 5^n + 2 \times 11^n$

Then $u_1 = 5 + 2 \times 11 = 27$ which is divisible by 3.

$\therefore$ **the result is true** for $n = 1$.

Technical issues

9 Prove by induction that $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers n. **[7]**

Suppose the result is true for $n = k$ for some $k \in \mathbb{N}$.

Then $u_k = 5^k + 2 \times 11^k = 3m$ from some $m \in \mathbb{Z}$

Consider

$u_{k+1} = 5^{k+1} + 2 \times 11^{k+1}$

$= 5 \times 5^k + 11 \times 2 \times 11^k$

$= 5(5^k + 2 \times 11^k) + 6 \times 2 \times 11^k$

$= 5 \times 3m + 3 \times 2 \times 2 \times 11^k$

$= 3\left(5m + 2^2 \times 11^k\right)$ which is a multiple of 3

So if u_k is divisible by 3, **then** u_{k+1} is

Conceptual issues

9 Prove by induction that $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers n. **[7]**

Therefore **the result is true for** $n = k + 1$ **if it is true for** $n = k$.

Since **it is true for** $n = 1$, by induction **it is true for** all positive integers n.

Which method?

Edexcel ASFM
Core Pure 2019

8. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

(6)

A. Assume $f(n)$ is divisible by 7 and show that $f(n+1)$ is?

B. Find $f(n+1) - f(n)$ and show it is divisible by 7?

C. Consider $f(n+1) - mf(n)$

8. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

Inductive Step:

Method 1

$$\begin{aligned} f(k+1) &= 2^{k+1+2} + 3^{2(k+1)+1} \\ &= 2^{k+3} + 3^{2k+3} \\ &= 2 \times 2^{k+2} + 9 \times 3^{2k+1} \\ &= 2\left(2^{k+2} + 3^{2k+1}\right) + 7 \times 3^{2k+1} \\ &= 2 \times 7m + 7 \times 3^{2k+1} \\ &= 7 \times (2m + 3^{2k+1}) \end{aligned}$$

8. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

Method 2

Inductive Step:

$$\begin{aligned} f(k+1) - f(k) &= 2^{k+1+2} + 3^{2(k+1)+1} - (2^{k+2} + 3^{2k+1}) \\ &= 2^{k+3} - 2^{k+2} + 3^{2k+3} - 3^{2k+1} \\ &= 2^{k+2}(2-1) + 3^{2k+1}(3^2 - 1) \\ &= 2^{k+2} + 8 \times 3^{2k+1} \\ &= 2^{k+2} + 3^{2k+1} + 7 \times 3^{2k+1} \\ &= f(k) + 7 \times 3^{2k+1} \end{aligned}$$

So $f(k+1) = 7 \times 3^{2k+1} + 2f(k)$ which is a multiple of 7 because…

8. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

Method 3

Inductive Step:

$$f(k+1) - mf(k) = 2^{k+1+2} + 3^{2(k+1)+1} - m(2^{k+2} + 3^{2k+1})$$

$$= 2^{k+3} - m \times 2^{k+2} + 3^{2k+3} - m \times 3^{2k+1}$$

$$= 2^{k+2}(2 - m) + 3^{2k+1}(3^2 - m)$$

$$= 2^{k+2}(2 - m) + 3^{2k+1}(7 + 2 - m)$$

$$= (2 - m)(2^{k+2} + 3^{2k+1}) + 7 \times 3^{2k+1}$$

$$= (2 - m)f(k) + 7 \times 3^{2k+1}$$

So $f(k+1) = 2f(k) + 7 \times 3^{2k+1}$ which is a multiple of 7 because…

Some top tips on one slide…

- Develop understanding - *not just rehearsal*
- Use precise language - *and make your students do the same*
- Make a convincing argument - *state the obvious!*
- Look at simple number theory - *separately from proof by induction*

Where next?

- *An Adventurer's Guide to Number Theory* by Richard Friedberg
- Project Euler (projecteuler.net) has lots of nice Number Theory problems
- Maths Item of the Month (https://mei.org.uk/month_item) has a number of proof by induction problems
- *All people in Canada are the same age* - a delightful fallacious proof online