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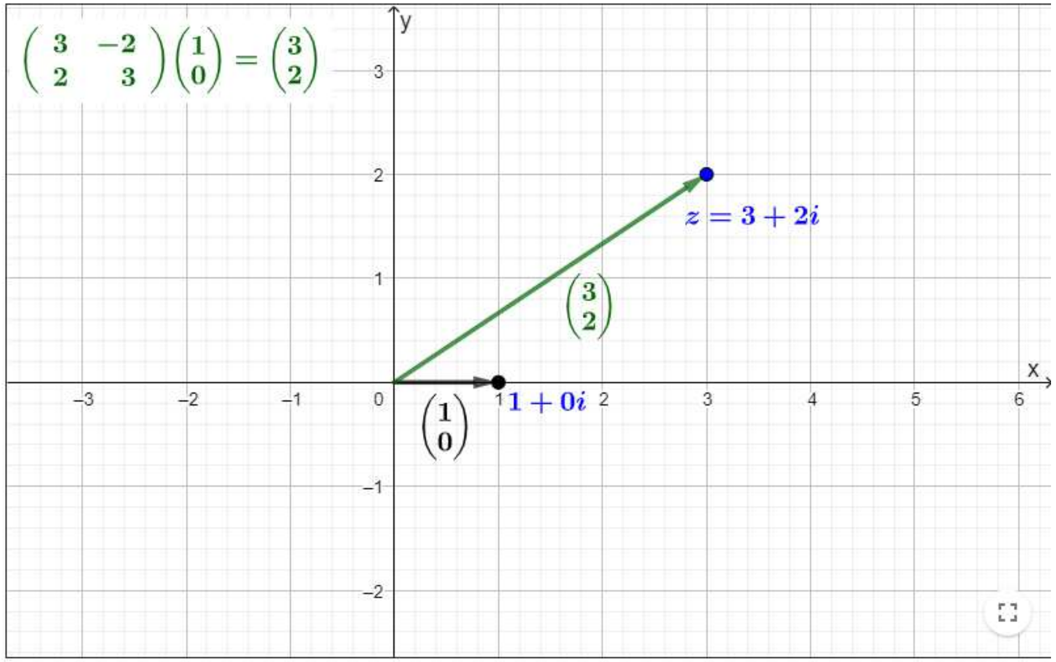

Linking complex numbers and matrices

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This session uses a GeoGebra activity:
www.geogebra.org/m/shyjdapc

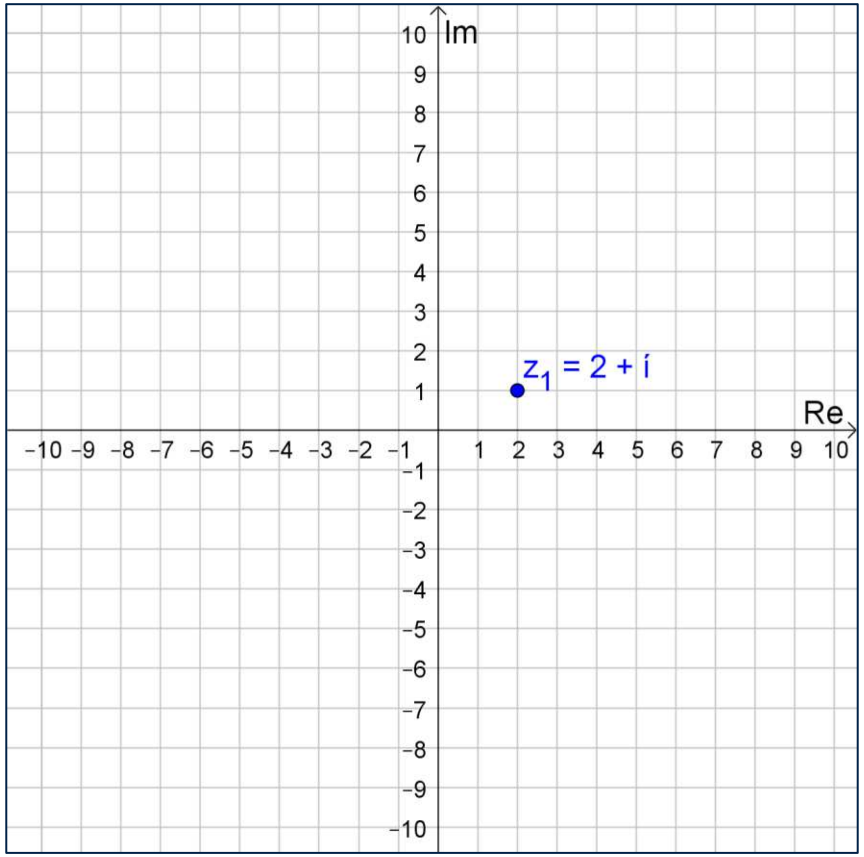
Format of the session

- This session is in a **Zoom** webinar
- You can message us privately using **Q&A**
- You can interact in the session via a ***GeoGebra Classroom Activity***
- The session **will be recorded**

Activity (a)

- $z_1 = 2 + \mathrm{i}$. Find examples of complex numbers, z_2, so that the product $z_1 z_2$ lies on the negative real axis.

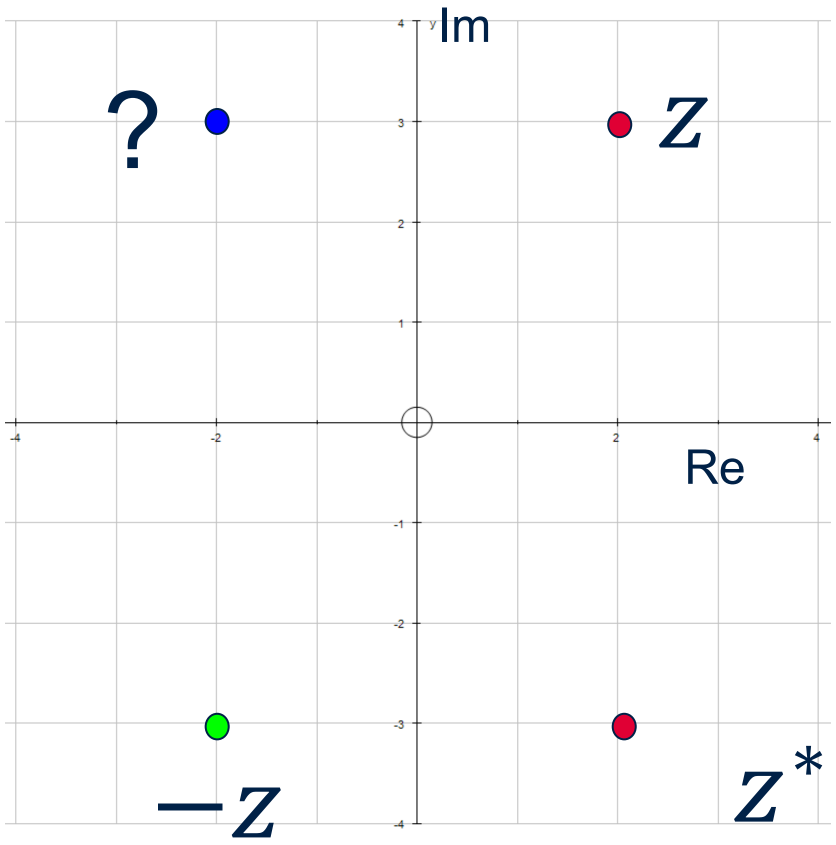
- Where do these complex numbers lie on an Argand diagram?



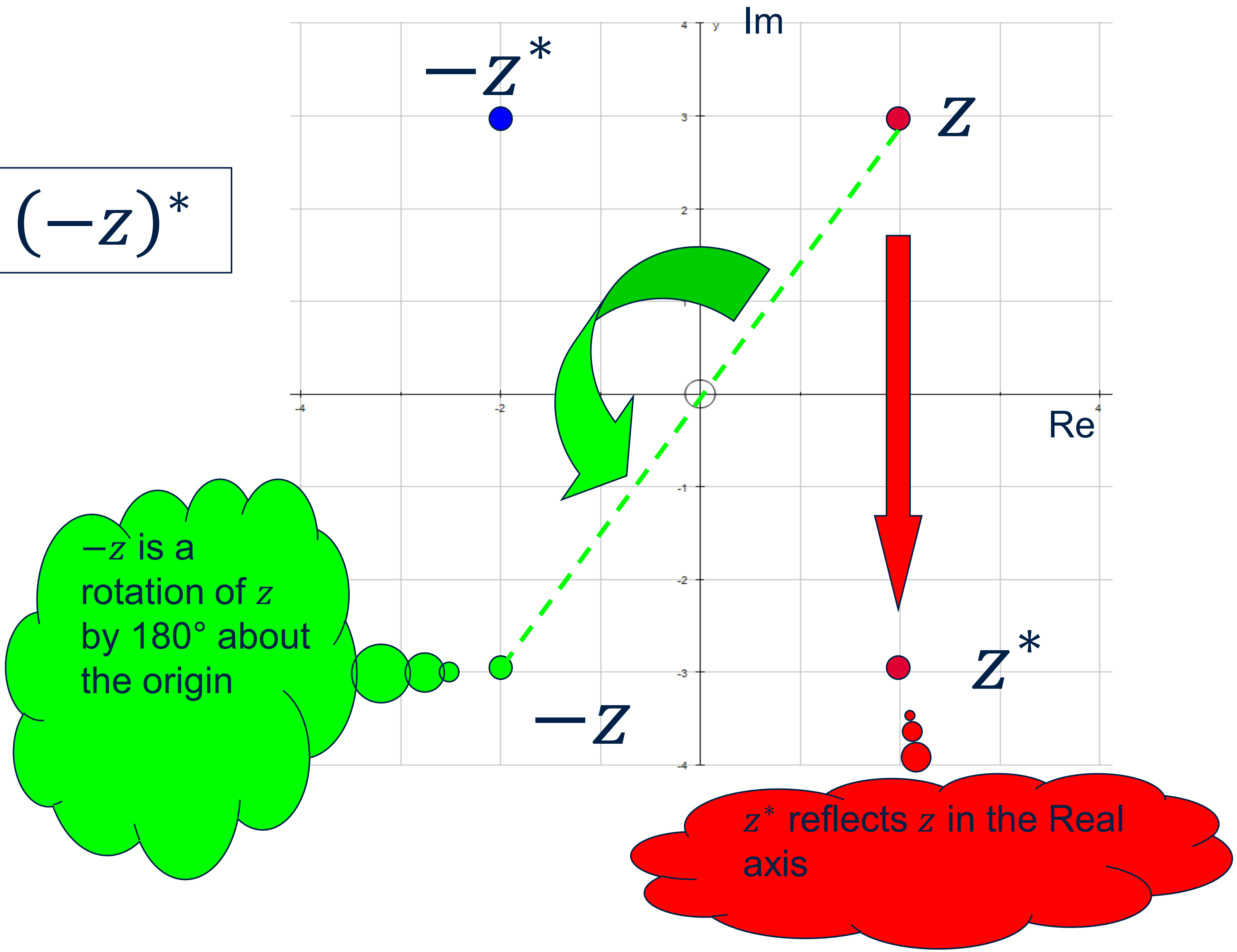
Argand diagram

In an Argand diagram, three vertices of a rectangle are given by the complex numbers z, z^* and $-z$.

What is the fourth vertex?



$$-(z^*) = (-z)^*$$



Im
Re
$-z^*$
z
$-z$
z^*
$-z$ is a rotation of z by 180° about the origin
z^* reflects z in the Real axis

Composite transformations

$$-(z^*)$$

Reflection in the real axis followed by a rotation 180°

$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

$$(-z)^*$$

Rotation 180° followed by a reflection in the real axis

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

Activity (b)

Activity (b) (i)

$w = 2 + \mathrm{i} \qquad z = 1 - 4\mathrm{i}$

Find:

- $w + z$
- wz
- $\frac{1}{w}$

Activity (b) (ii)

$\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$

Find:

- $\mathbf{M} + \mathbf{N}$
- $\mathbf{MN}$
- $\mathbf{M}^{-1}$

Answers to activity (b)

Activity 2A

$w = 2 + \mathrm{i} \qquad z = 1 - 4\mathrm{i}$

$2 + \mathrm{i} + 1 - 4\mathrm{i} = 3 - 3\mathrm{i}$

$(2 + \mathrm{i})(1 - 4\mathrm{i}) = 6 - 7\mathrm{i}$

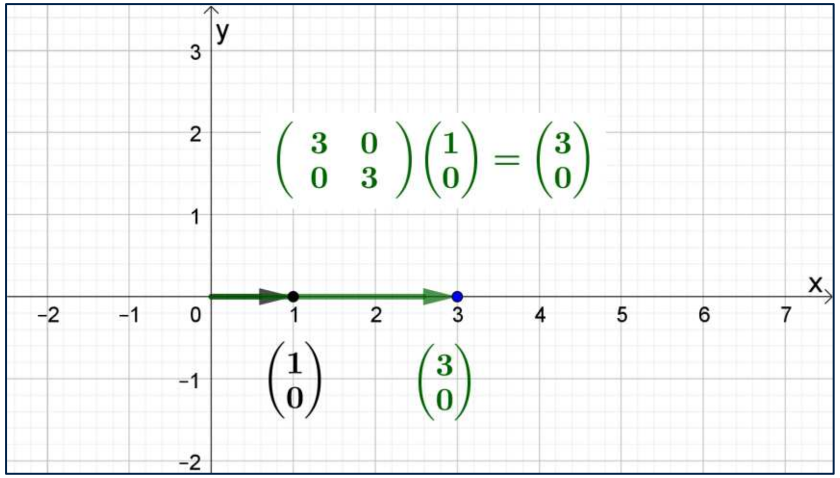
Activity 2B

$\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -3 & 3 \end{pmatrix}$

$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ -7 & 6 \end{pmatrix}$

Numbers as enlargements of 1



$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \qquad 2 + 3 = 5$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \qquad 2 \times 3 = 6$$

$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ behaves in the same way as the real number a

Activity (c): Finding a square root of –1

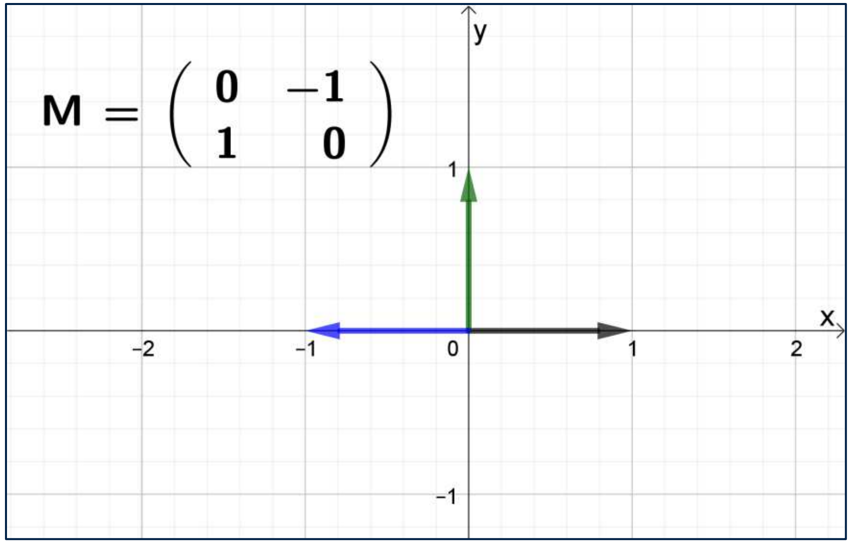
Find a matrix $\mathbf{M}$ (with real elements) such that

$$\mathbf{M}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Finding a square root of -1

$$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$$ or $$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$$

$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ can represent i

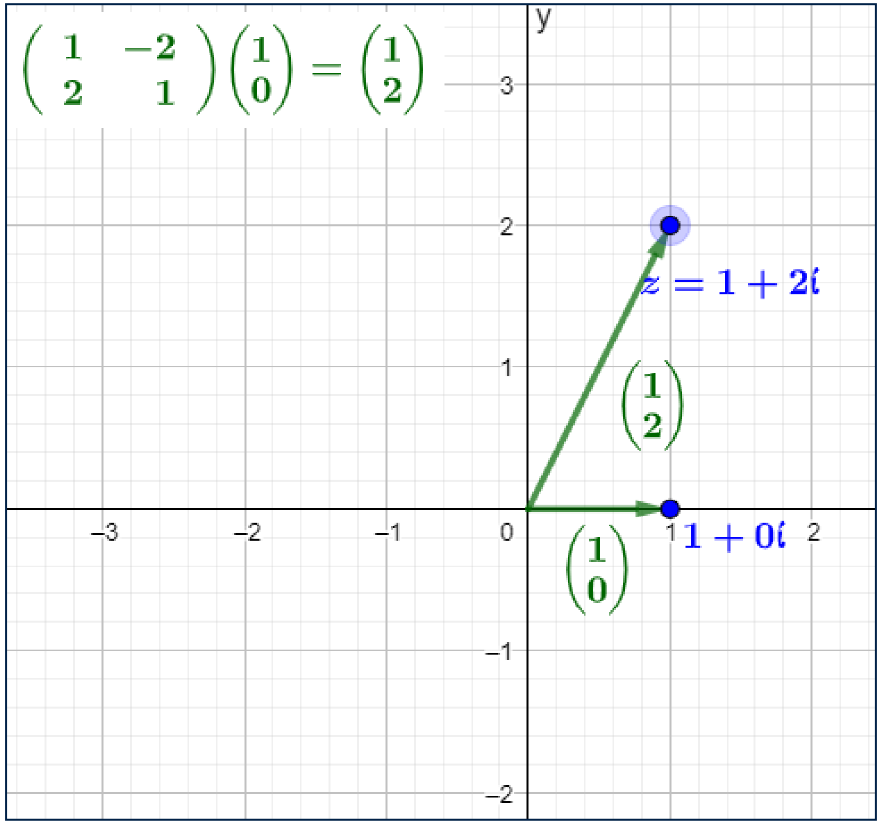


Representing a complex number as a matrix

$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ represents the real number a

$\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$ represents the imaginary number bi

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ represents the complex number $a + bi$



www.geogebra.org/m/suurfhps

Activity (b) recap

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ represents the complex number $a + b\mathrm{i}$

Activity 2A

$w = 2 + \mathrm{i}$ $\qquad$ $z = 1 - 4\mathrm{i}$

$2 + \mathrm{i} + 1 - 4\mathrm{i} = 3 - 3\mathrm{i}$

$(2 + \mathrm{i})(1 - 4\mathrm{i}) = 6 - 7\mathrm{i}$

Activity 2B

$\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ $\qquad$ $\mathbf{N} = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -3 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ -7 & 6 \end{pmatrix}$$

Activity (d)

Solve the quadratic equation $z^2 - 4z + 13 = 0$.

Write z as a matrix $\mathbf{M}$.

Find the matrices $\mathbf{M}^2$, $4\mathbf{M}$ and $13\mathbf{I}$ and hence show that $\mathbf{M}^2 - 4\mathbf{M} + 13\mathbf{I} = \mathbf{0}$.

Activity (e) (extension)

Find the complex number $z = 3 + 4\text{i}$ in modulus argument form.

Write z as a matrix $\mathbf{M}$.

Show that the matrix $\mathbf{M}$ can be written as $r\begin{pmatrix}\sin\theta & -\cos\theta \\ \cos\theta & \sin\theta\end{pmatrix}$.

Is this true for any matrix of the form $\begin{pmatrix}a & -b \\ b & a\end{pmatrix}$?

Where's the flaw?

$$\begin{aligned} 1 &= \sqrt{1} \\ &= \sqrt{-1 \times -1} \\ &= \sqrt{-1} \times \sqrt{-1} \\ &= \mathrm{i} \times \mathrm{i} \\ &= -1 \end{aligned}$$

Find the roots of an equation

~~Find $\sqrt{-9}$~~

Find the roots of the equation $z^2 = -9$

Writing this when solving a quadratic equation is acceptable as the $\pm$ sign indicates both roots. → $z = \frac{-4 \pm \sqrt{-36}}{2}$

Thank you!

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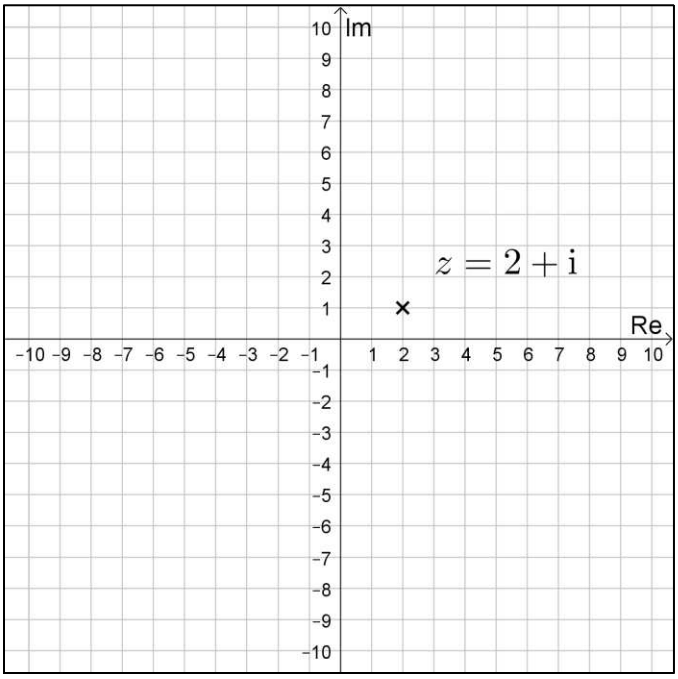
Linking complex numbers and matrices

Activity (a)

$z = 2 + \mathrm{i}$. Find examples of complex numbers, w, so that the product wz lies on the negative real axis.

Plot these on an Argand diagram.



Activity (b) (i)

$w = 2 + \mathrm{i}$ $\qquad$ $z = 1 - 4\mathrm{i}$

Find:

- $w + z$
- wz
- $\frac{1}{w}$

Activity (b) (ii)

$\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ $\qquad$ $\mathbf{N} = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$

Find:

- $\mathbf{M} + \mathbf{N}$
- $\mathbf{MN}$
- $\mathbf{M}^{-1}$

Activity (c)

Find a 2×2 matrix $\mathbf{M}$ (with real elements) such that $\mathbf{M}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Activity (d)

(i) Solve the quadratic equation $z^2 - 4z + 13 = 0$.

Hence find a matrix $\mathbf{M}$ such that $\mathbf{M}^2 - 4\mathbf{M} + 13\mathbf{I} = \mathbf{0}$. Verify this by calculating $\mathbf{M}^2$, $4\mathbf{M}$ and $13\mathbf{I}$.

Activity (e)

(ii) Find the complex number $z = 3 + 4\text{i}$ in modulus argument form.

Show that the matrix $\mathbf{M} = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$ can be written as $r\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$.

Is this true for any matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$?