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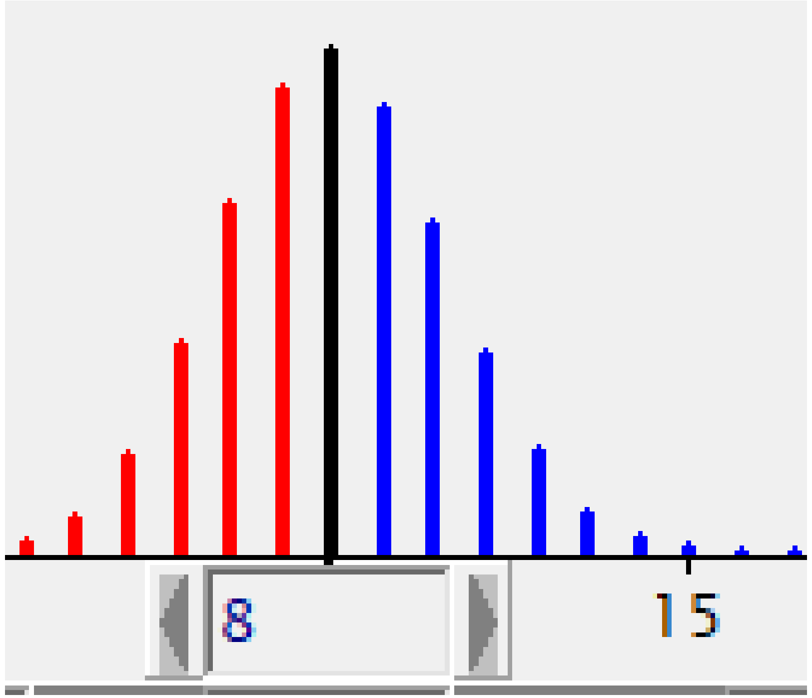

Continuing Professional Development Standard
National Centre for Excellence in the Teaching of Mathematics

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Using Probability Generating Functions

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Probability Generating Function

A way of representing a probability distribution in a compact form, as a function, that allows key results to be derived in a fairly straightforward way and allows connections to be made.

Representing Probability Distributions

Consider throwing a fair four-sided dice.

There are different ways that we can represent the possible outcomes and their respective probabilities.

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As a distribution function: $P(X = x) = \frac{1}{4} \qquad x = 1, 2, 3, 4$

Probability Generating Function

However there is another representation which may seem a little strange at first

x	1	2	3	4
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

And that is to form a polynomial whose coefficients are the probabilities and whose powers are the values of x.

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$\frac{1}{4}t+\frac{1}{4}t^2+\frac{1}{4}t^3+\frac{1}{4}t^4$ This is usually referred to as $G_X(t)$.

Probability Generating Function

More generally we have:

$$G_X(t) = \sum P(X = x)t^x$$

summed over all values of the random variable which have non-zero probabilities.

These functions are useful as they contain all the information about the distribution in one function.

G(1)

$$G_X(t) = \sum P(X = x)t^x$$

What is $G(1)$?

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Screen 2

G(1)

$$G_X(t) = \sum P(X = x)t^x$$

$$G(1) = \sum P(X = x) \times 1^x = \sum P(X = x)$$

G(1)

$$G_X(t) = \sum P(X = x)t^x$$

$$G(1) = \sum P(X = x) \times 1^x = \sum P(X = x) = 1$$

Examples

$$G_X(t) = 0.3 + 0.1t + 0.2t^2 + 0.2t^3 + 0.1t^4 + 0.05t^5 + 0.05t^6$$

How do we know that this is a valid PGF?

What values can the random variable X take?

What is $P(X = 0)$ $P(X = 4)$ $P(X = 7)$?

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Examples

$$G_X(t) = k(1+t)^8$$

Find k.

What is the largest value X can take?

$$G_X(t) = \sum P(X=x)t^x$$

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Screen 4

$$G(1) = \sum P(X=x) \times 1^x = \sum P(X=x) = 1$$

Examples

$$G_X(t) = \frac{1}{256}(1+t)^8$$

Find $P(X = 4)$

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The term in t^4 is $\frac{1}{256} \times \binom{8}{4} 1^4 t^4$

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Find $P(X = 4)$

The term in t^4 is $\frac{1}{256} \times \binom{8}{4} 1^4 t^4$

$$P(X = 4) = \frac{1}{256} \times \frac{8!}{4!\,4!} = \frac{35}{128}$$

Examples

$$G_X(t) = \frac{1}{256}(1+t)^8$$

What is the distribution of X?

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Examples

$$G_X(t) = (0.5 + 0.5t)^8$$

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$$G_X(t) = (0.5 + 0.5t)^8$$
$$= 0.5^8 + \binom{8}{1}(0.5)^7(0.5t)^1 + \binom{8}{2}(0.5)^6(0.5t)^2 + \cdots$$

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What is the distribution of X?

$X \sim B(8,0.5)$

Binomial distribution

More generally for a binomial distribution:

$$G_X(t) = (1 - p + pt)^n$$
$$= (1-p)^n + \binom{n}{1}(1-p)^{n-1}(pt) + \binom{n}{2}(1-p)^{n-2}(pt)^2 + \cdots$$

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The term in t^x is $\binom{n}{x}(1-p)^{n-x}(pt)^x$

The coefficient of t^x is $\binom{n}{x}(1-p)^{n-x}p^x$ which is $P(X = x)$

Poisson distribution

A Poisson distribution is used to model events that occur randomly and singly in time or space at fixed rate.

It has probability distribution function:

$P(X = x) = \frac{e^{-\mu} \times \mu^x}{x!}$ where μ is the rate of occurrence in a given interval.

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What is the PGF?

$$G_X(t) = \sum_{x=0}^{\infty} \frac{e^{-\mu} \times \mu^x}{x!} \times t^x$$

Poisson distribution

What is the PGF?

$$G_X(t) = \sum_{x=0}^{\infty} \frac{e^{-\mu} \times \mu^x}{x!} \times t^x = e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu t)^x}{x!}$$

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$$= e^{-\mu} e^{\mu t} = e^{\mu(t-1)}$$

Some properties

$$G_X(t) = \sum P(X = x)t^x$$

What is $G'_X(t)$?

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$$G'_X(1) = \sum xP(X = x)$$

What is this?

Some properties

$$G_X(t) = \sum P(X = x)t^x$$

What is $G'_X(t)$?

$$G'_X(t) = \sum xP(X = x)t^{x-1}$$

$$G'_X(1) = \sum xP(X = x) = E(X)$$

E(X)

$E(X) = G'_X(1)$

For a binomial distribution:

$G_X(t) = (1 - p + pt)^n$

For a Poisson distribution:

$G_X(t) = e^{\mu(t-1)}$

Can you find $E(X)$ for both of these?

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E(X)

$$G_X'(1) = \sum xP(X = x) = E(X)$$

For a binomial distribution:

$$G_X(t) = (1 - p + pt)^n$$
$$G_X'(t) = np(1 - p + pt)^{n-1}$$
$$G_X'(1) = np(1 - p + p)^{n-1} = np$$

E(X)

$$G'_X(1) = \sum xP(X = x) = E(X)$$

For a Poisson distribution:

$$G_X(t) = e^{\mu(t-1)}$$

$$G'_X(t) = \mu e^{\mu(t-1)}$$

$$G'_X(1) = \mu e^0 = \mu$$

Using PGFs

If we can show that two random variables have the same PGF then we have shown that the two random variables have the same distribution.

The PGF of X tells us everything there is to know about the distribution of X.

Using PGFs

The power of the PGF is that it gives us an easy way of determining the distribution of the sum of two random variables $X + Y$ when X and Y are independent.

This is difficult to do using probability distribution functions.

The PGF transforms a sum into a product which enables it to be manipulated more readily.

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$G_{X+Y}(t) = E(t^{X+Y}) = E(t^X t^Y) = E(t^X)E(t^Y)$ as $X\ and\ Y$ independent

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So consider two independent random variables $X\ and\ Y$ with PGFs $G_X(t)$ and $G_Y(t)$ respectively.

$$\boldsymbol{G_{X+Y}(t) = G_X(t)G_Y(t)}$$

The sum of 2 Poisson variables

Suppose that X and Y are independent with $X \sim$ Poisson(λ) and $Y \sim$ Poisson(μ).

Find the distribution of $X + Y$

For a Poisson distribution:
$G_X(t) = e^{\mu(t-1)}$

The sum of 2 Poisson variables

Suppose that X and Y are independent with $X \sim$ Poisson(λ) and $Y \sim$ Poisson(μ).

Find the distribution of $X + Y$

$$G_{X+Y}(t) = G_X(t)G_Y(t) = e^{\lambda(t-1)}e^{\mu(t-1)}$$

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The sum of 2 Poisson variables

Suppose that X and Y are independent with $X \sim$ Poisson(λ) and $Y \sim$ Poisson(μ).

Find the distribution of $X + Y$

$$G_{X+Y}(t) = G_X(t)G_Y(t) = e^{\lambda(t-1)}e^{\mu(t-1)} = e^{(\lambda+\mu)(t-1)}$$

which is a Poisson distribution with parameter $\lambda + \mu$.

For a Poisson distribution:
$G_X(t) = e^{\mu(t-1)}$

The geometric and negative binomial distributions

Imagine rolling a six-sided dice where getting a six on a single throw has probability p

How many rolls are needed to get the first six?

How many rolls are needed to get the r^{th} six?

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Geometric distribution $P(X = x) = p(1-p)^{x-1}$

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$$G_X(t) = \sum p(1-p)^{x-1}t^x = pt\sum (1-p)^{x-1}t^{x-1}$$

$$= pt\sum (t(1-p))^{x-1}$$

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Negative binomial $P(Y = y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$

Using a similar argument:

$$G_Y(t) = \left(\frac{pt}{1-(1-p)t}\right)^r$$

The geometric and negative binomial distributions

So if take r independent observations of this geometric variable and sum them:

$Y = X_1 + X_2 + \cdots + X_r$

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So $G_Y(t) = G_{X_1}(t)G_{X_2}(t)\ldots\ldots G_{X_r}(t)$

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But these are all equal to $G_X(t)$

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$$G_Y(t) = (G_X(t))^r = \left(\frac{pt}{1-(1-p)t}\right)^r$$

But this is the PGF for a negative binomial.

In summary

- $G_X(t) = \sum P(X = x)t^x$ is an alternative way of writing a probability distribution.
- It can be thought of as $E(t^X)$
- $G(1) = \sum P(X = x) = 1$
- $E(X) = G_X'(1)$
- $Var(X) = G_X''(1) + G_X'(1) - (G_X'(1))^2$
- The common distributions have fairly ‘compact’ PGFs – can be reasonably straightforward to work out $E(X)$ and $Var(X)$
- $G_{X+Y}(t) = G_X(t)G_Y(t)$ if X and Y are independent – this helps to derive the distribution of a sum of random variables.

Other uses

- Expectation and variance of functions of random variables
- Sums of random variables where the number of variables is itself random (a randomly stopped sum)
- Random walks (expected reach times)
- Deciding whether a process will reach a given state.
- Sicherman dice: session I4 at https://mei.org.uk/conference13 .