Advanced Mathematics Support Programme®

The problem-solving skills required for success in admissions tests

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Desmos activities at
student.desmos.com
Code: 57H86E

Continuing Professional Development Standard

National Centre for Excellence in the Teaching of Mathematics

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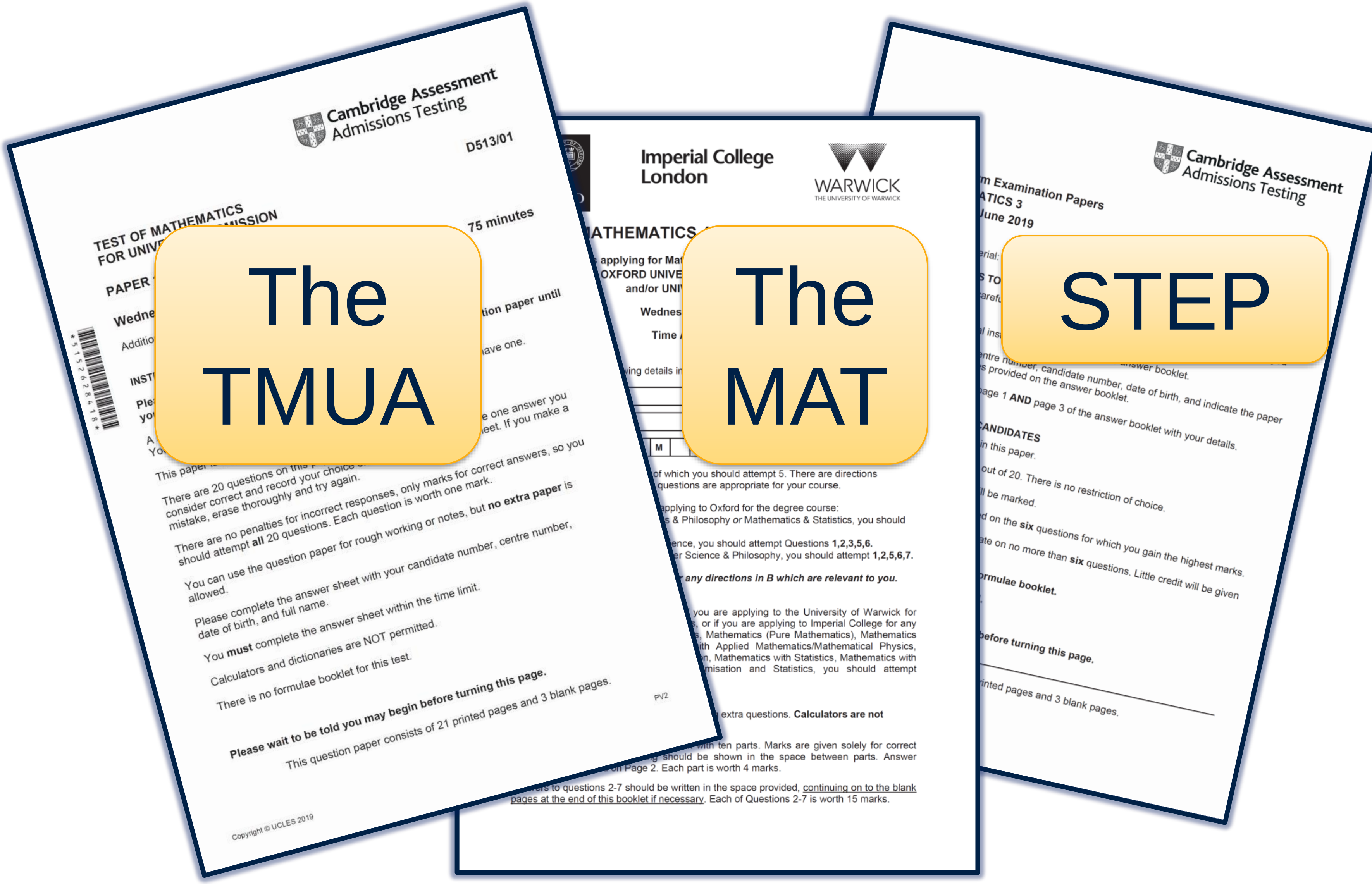

A suggested definition of problem solving

- A student has to engage in problem-solving in mathematics when they are faced with something that they do not immediately recognise as a version of a something they have solved before.

- As a consequence they must devise a strategy for solving the problem themselves, and exploration may be needed to find it, rather than simply using a strategy already known to them.

The TMUA

- The TMUA is designed to give you the opportunity to demonstrate that you have the essential mathematical thinking and reasoning skills needed for a demanding undergraduate Mathematics or Mathematics-related course.

The TMUA

- Focuses on assessing your ability to apply your knowledge of mathematics in new situations.

- Focuses on assessing your ability to deal with mathematical reasoning, and simple ideas from elementary logic.

The MAT

- The MAT aims to test the depth of mathematical understanding of a student in the fourth term of their A-levels (or equivalent)

- rather than a breadth of knowledge.

The MAT

- It is set with the aim of being approachable by all students, including those without Further Mathematics A-level, and those from other educational systems (e.g. Baccalaureate and Scottish Highers).

STEP

- Questions may test a candidate’s ability to apply mathematical knowledge in novel and unfamiliar ways

- will often require knowledge of several different specification topics.

STEP

- Solutions will frequently require insight, ingenuity, persistence

- the ability to work through substantial sequences of algebraic manipulation.

A Warm-Up Question

- MAT style

p and q are positive integers such that $p + q = 13$.

What is the maximum value that $p^3 q$ can take?

a) 2662 b) 2916 c) 3000

d) 3048 c) 3401

p and q are positive integers such that $p + q = 13$.

What is the maximum value that p^3q can take?

p and q are positive integers such that $p + q = 13$.

What is the maximum value that p^3q can take?

Is this important?

p and q are positive integers such that $p + q = 13$.

What is the maximum value that p^3q can take?

How does that tie in with this?

Is there any way to represent this?

p and q are positive integers such that $p + q = 13$.

What is the maximum value that p^3q can take?

What does this word suggest?

p and q are positive integers such that $p + q = 13$.

What is the maximum value that p^3q can take?

How can you find the maximum of this?

Try it out

- Try to solve the question
- Can you do it in under 4 minutes?

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Presenter Warning!

The answers start on the next slide

$q = 13 - p$

$V = p^3 q = p^3(13 - p)$
$V = 13p^3 - p^4$

$$\frac{\mathrm{d}V}{\mathrm{d}p} = 39p^2 - 4p^3$$

$39p^2 - 4p^3 = 0$

$p^2(39 - 4p) = 0$

$$p = \frac{39}{4}$$

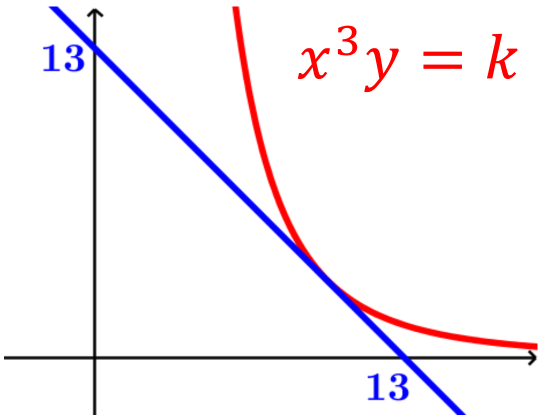
p is a positive integer

$p = 9$ or $p = 10$

$V = p^3(13 - p)$

$p = 9, V = 729 \times 4 = 2916$

$p = 10, V = 1000 \times 3 = 3000$



$x^3y = k \Leftrightarrow y = \frac{k}{x^3}$

$x + y = 13$

$$x + \frac{k}{x^3} = 13 \qquad *$$

When the graphs touch, the gradient is –1 for both

$y = kx^{-3}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -3kx^{-4}$

$$-\frac{3k}{x^4} = -1 \Leftrightarrow k = \frac{x^4}{3}$$

In * $\quad x + \frac{x}{3} = 13 \Leftrightarrow \frac{4x}{3} = 13$

$$x = \frac{39}{4}$$

so use $p = 9$ or $p = 10$

$$p^3q = 9^3 \times 4 = 2916$$

$$p^3q = 10^3 \times 3 = 3000$$

The TMUA specification

- **SECTION 1**

- The content of Part 1 is almost all covered within the pure mathematics specification of an AS level in mathematics, and the content of Part 2 is almost all covered within a Higher-Level GCSE mathematics course.

The TMUA specification

- **SECTION 2**

- Paper 2 tests the candidate's ability to think mathematically: the paper will focus on testing the candidate's ability to understand, and construct, mathematical arguments in a variety of contexts. It will draw on the mathematical knowledge outlined in SECTION 1

Problem Solving in the TMUA

- You have 4 TMUA questions
- These come from both paper 1 and paper 2
- Read through the questions
- Attempt them but think…
  - What's the difference between this and an A level question?
  - **What "extra" would my students need to succeed?**

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Activity code: 57H86E

A. Find the value of

$$\sin^2 0° + \sin^2 1° + \sin^2 2° + \sin^2 3° + \cdots$$
$$+ \sin^2 87° + \sin^2 88° + \sin^2 89° + \sin^2 90°$$

A	0.5	B	1	C	1.5
D	45	E	45.5	F	46

2018 P1 Q20

$\sin^2 0° + \sin^2 90° \equiv \sin^2 0° + \cos^2 (90-90)°$

$\sin^2 1° + \sin^2 89° \equiv \sin^2 1° + \cos^2 1°$

45 pairs that sum to 1 and $\sin^2 45°$

$45 + \left(\frac{\sqrt{2}}{2}\right)^2 = 45 + \frac{1}{2} = 45.5$ (E)

B. The sequence of functions

$\mathrm{f}_1(x), \mathrm{f}_2(x), \mathrm{f}_3(x), \ldots$

is defined as follows

$\mathrm{f}_1(x) = x^{10}$, $\mathrm{f}_{n+1}(x) = x\mathrm{f}'_n(x)$ for $n \geq 1$

where $\mathrm{f}'_n(x) = \dfrac{\mathrm{d}\mathrm{f}_n(x)}{\mathrm{d}x}$

Find the value of $\displaystyle\sum_{n=1}^{20} \mathrm{f}_n(x)$

2016 P2 Q6

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$f_1(x) = x^{10}$

$f_2(x) = x \cdot 10x^9 = 10x^{10}$

$f_3(x) = x \cdot 100x^9 = 100x^{10}$

$f_4(x) = 1000x^{10}$

$f_n(x) = 10^{n-1}x^{10}$

$a = x^{10} \quad r = 10 \quad n = 20$

$\sum_{n=1}^{20} f_n(x) = S_{20}$ for a geometric series

$= \frac{x^{10}(10^{20} - 1)}{9}$ (C)

C. The two diagonals of the quadrilateral Q are perpendicular.

Consider the following statements:

I One of the diagonals of Q is a line of symmetry of Q.

II The midpoints of the sides of Q are the vertices of a square.

Which of these statements is/are **necessarily** true for a quadrilateral Q?

A neither of them

B I only

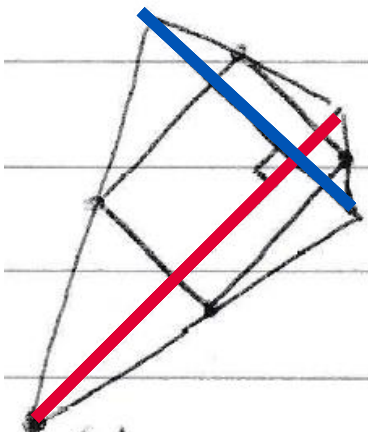
C II only

D I and II

2018 P2 Q5

I is not true (A)

II is not true



↙ imagine lengthening the diagonal by moving this vertex

D. I have forgotten my 5-character computer password but I know that it consists of the letters **a**, **b**, **c**, **d**, **e** in some order. When I enter a potential password into the computer, it tells me exactly how many of the letters are in the correct position.

When I enter **abcde**, it tells me that none of the letters are in the correct position. The same happens when I enter **cdbea** and **eadbc**.

Using the best strategy, how many **further** attempts must I make in order to **guarantee** that I can **deduce** the correct password?

A None: I can deduce it immediately

B One

C Two

D Three

E More than three

2017 P2 Q20

1	a	c	e	b/d
2	b	d	a	c/e
3	c	b	d	a/e
4	d	e	b	a/c
5	e	a	c	b/d

Rows 2, 3, 4: one of these gives the other two

Rows 1–5: one of these gives the other

Needs <u>at least</u> one more try

If 0 correct then know password (just swap all)

Can't have just 1 correct

If 2 correct it must be 1 & 5 so swap the others

If 3 correct it must be 2, 3 & 4 so swap 1 and 5

Can't have 4 correct

If 5 correct → then know the password

(B)

The MAT syllabus

- Polynomials
- Algebra
- Differentiation
- Integration
- Graphs
- Logarithms and powers
- Transformations

The MAT syllabus

- Geometry
- Trigonometry (using degrees)
- Sequences and series

Problem Solving in the MAT

- You have 5 MAT questions
- The first 4 are multiple choice, the last one is one of the longer questions
- Read through the questions
- Attempt them but think…
    - What's the difference between this and an A level question?
    - What "extra" would my students need to succeed?

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Activity code: 57H86E

A. The function $y = e^{kx}$ satisfies the equation

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}-y\right)=y\frac{\mathrm{d}y}{\mathrm{d}x}$$

for

(a) no values of k,

(b) exactly one value of k,

(c) exactly two distinct values of k,

(d) exactly three distinct values of k,

(e) Infinitely many distinct values of k.

2018 Q1B

$$y = e^{kx} \qquad \frac{dy}{dx} = ke^{kx} \qquad \frac{d^2y}{dx^2} = k^2e^{kx}$$

$$(k^2e^{kx} + ke^{kx})(ke^{kx} - e^{kx}) = e^{kx} . ke^{kx}$$

$$e^{kx}(k^2 + k)e^{kx}(k - 1) = ke^{kx} . e^{kx}$$

$$e^{kx} \neq 0$$

$$(k^2 + k)(k - 1) = k$$

$$k(k+1)(k-1) = k$$

$$k(k^2 - 1) - k = 0$$

$$k(k^2 - 1 - 1) = 0$$

$$k(k^2 - 2) = 0$$

$$0, \pm\sqrt{2} \qquad \text{(d)}$$

B. The product of a square number and a cube number is

(a) always a square number, and never a cube number.

(b) always a cube number, and never a square number.

(c) sometimes a square number, and sometimes a cube number.

(d) never a square number, and never a cube number.

(e) always a cube number, and always a square number.

2019 Q1B

Counter examples help

1 is both a square and a cube $1 \times 1 = 1$

a) Not true b) not true, c) is true

d) not true e) not true

↓

held until d, e checked

C. An equilateral triangle has centre O and side length 1. A straight line through O intersects the triangle at two distinct points P and Q. The minimum possible length of PQ is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{3}$ (d) $\frac{2}{3}$ (e) $\frac{\sqrt{3}}{2}$

2019 Q1J

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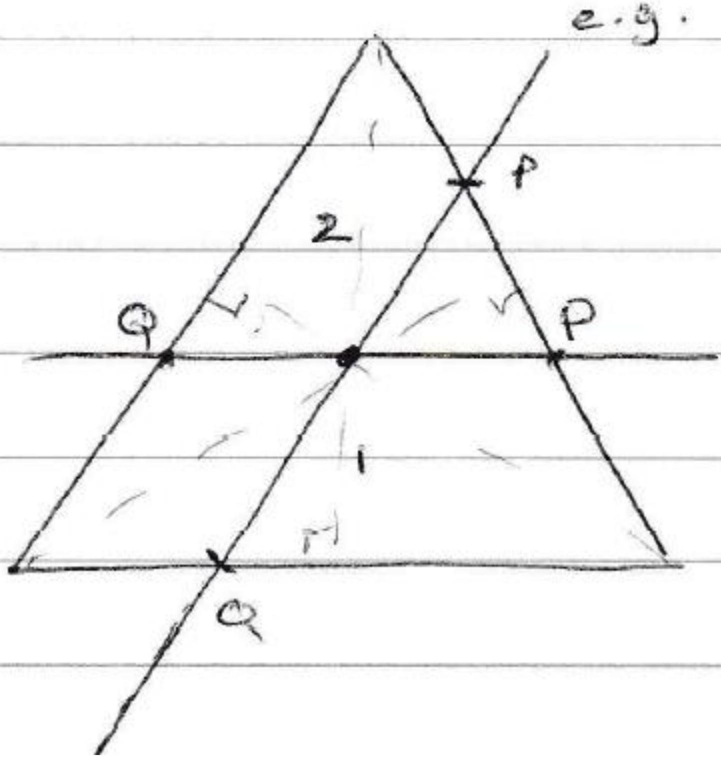


2 : 1 ratio

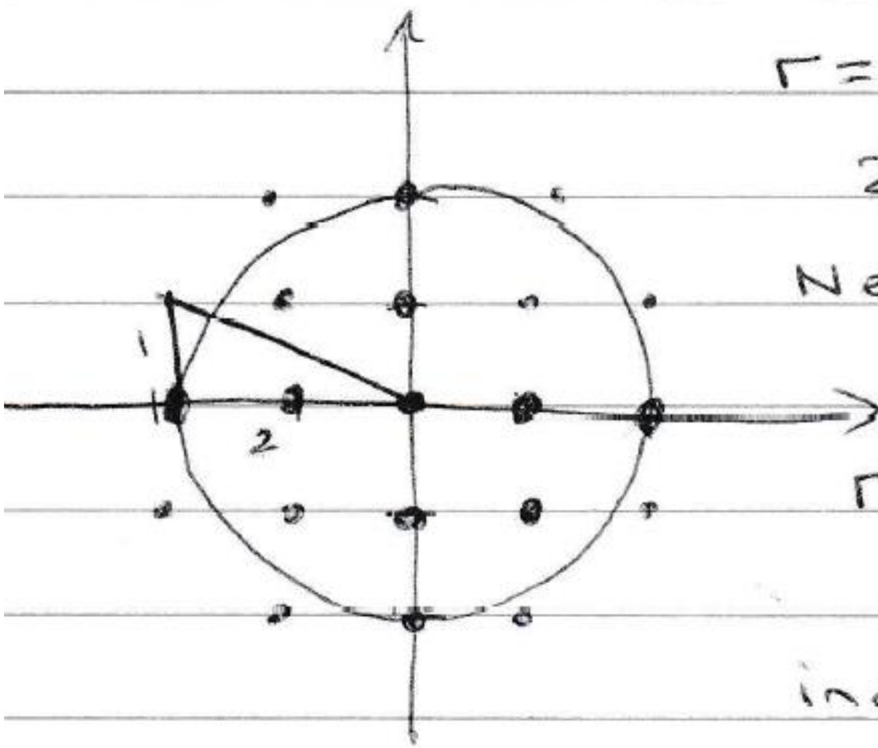
$\Rightarrow PQ = \frac{2}{3}$ (d)

D. A circle of radius 2, centered on the origin, is drawn on a grid of points with integer coordinates.

Let n be the number of grid points that lie within or on the circle. What is the smallest amount the radius needs to increase by for there to be $2n - 5$ grid points within or on the circle?

(a) $\sqrt{5} - 2$ (b) $\sqrt{6} - 2$ (c) $\sqrt{8} - 2$ (d) 1 (e) $\sqrt{8}$

2018 Q1E



$r = 2 \quad n = 13$

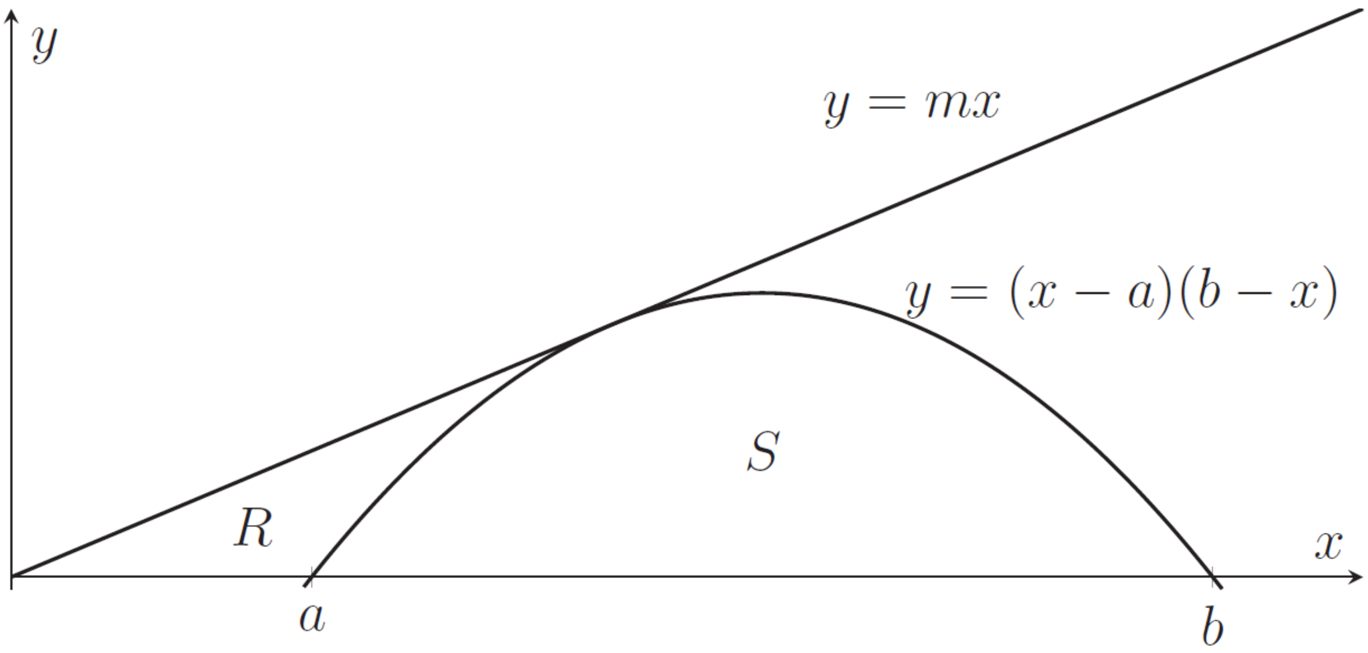
$2n - 5 = 21$

Need 8 more points

$r = \sqrt{5}$

increase $= \sqrt{5} - 2 \Rightarrow$ (a)

E. Let a, b, m be positive numbers with $0 < a < b$. In the diagram below are sketched the parabola with equation $y = (x - a)(b - x)$ and the line $y = mx$. The line is tangential to the parabola.
R is the region bounded by the x-axis, the line and the parabola.
S is the region bounded by the parabola and the x-axis.



(i) For $c > 0$, evaluate

$$\int_0^c x(c-x)\ \mathrm{d}x.$$

Without further calculation, explain why the area of region S equals

$$\frac{(b-a)^3}{6}$$

(i) $\int_0^c (cx - x^2)\,dx = \left[\frac{1}{2}cx^2 - \frac{1}{3}x^3\right]_0^c$

$= \frac{1}{2}c^3 - \frac{1}{3}c^3 = \frac{1}{6}c^3$ *

Since $b - a > 0$

Let $c = b - a$

The region calculated by * is a translation $\begin{pmatrix}-a\\0\end{pmatrix}$

of S and so has the same area i.e. $\frac{(b-a)^3}{6}$

(ii) The line $y = mx$ meets the parabola tangentially as drawn in the diagram.

Show that $m = \left(\sqrt{b} - \sqrt{a}\right)^2$.

(ii) $mx = (x-a)(b-x)$ has a repeated root

$mx = -x^2 + (a+b)x - ab$

$x^2 + (m-a-b)x + ab = 0$ has a repeated root

$\Rightarrow \Delta = 0 \qquad (m-a-b)^2 - 4ab = 0$

$m - a - b = \pm 2\sqrt{ab}$

$m = a + b \pm 2\sqrt{ab}$

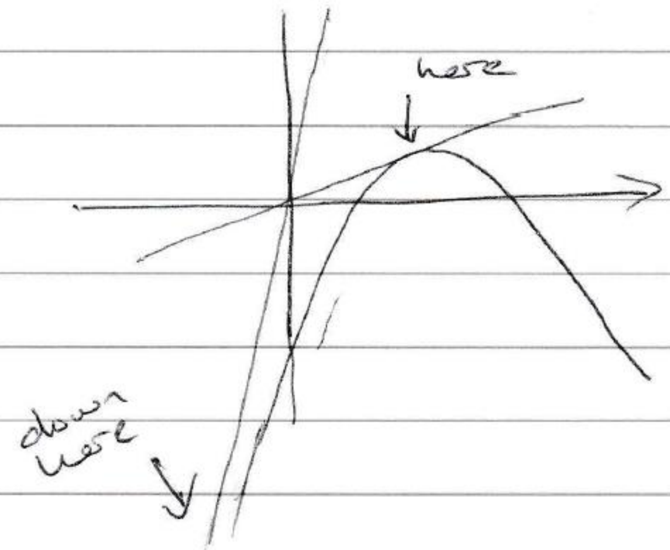
$(\sqrt{b} + \sqrt{a})^2 = a + b + 2\sqrt{ab}$

$(\sqrt{b} - \sqrt{a})^2 = a + b - 2\sqrt{ab}$

$m = (\sqrt{b} \pm \sqrt{a})^2$

There are two possible tangents

$(\sqrt{b}-\sqrt{a})^2 < (\sqrt{b}+\sqrt{a})^2$



Want the one in quadrant 1 so $(\sqrt{b}-\sqrt{a})^2$

(iii) Assume now that $a = 1$ and write $b = \beta^2$ where $\beta > 1$.

Given that the area of R equals $(2\beta + 1)(\beta - 1)^2/6$, show that the areas of regions R and S are equal precisely when

$$(\beta - 1)^2(\beta^4 + 2\beta^3 - 4\beta - 2) = 0. \quad (*)$$

Explain why there is a solution β to $(*)$ in the range $\beta > 1$.

Without further calculation, deduce that for any $a > 0$ there exists $b > a$ such that the area of region S equals the area of region R.

2019 Q3

(iii) $a=1 \quad b=\beta^2$

$R=S \quad \dfrac{(2\beta+1)(\beta-1)^2}{6} = \dfrac{(\beta^2-1)^3}{6}$

$(\beta-1)^2(2\beta+1) = (\beta-1)^2(\beta^2-1)(\beta+1)^2$

$(\beta-1)^2\left[(\beta^2-1)(\beta^2+2\beta+1)-2\beta-1\right]=0$

$(\beta-1)^2\left[\beta^4+2\beta^3+\beta^2-\beta^2-2\beta-1-2\beta-1\right]=0$

$(\beta-1)^2(\beta^4+2\beta^3-4\beta-2)=0$

$(\beta-1)^2$ always gives $\beta=1$ so for $\beta>1$ to give equal areas, the value of β must come from the factor $\beta^4+2\beta^3-4\beta-2$

Let $f(x)=x^4+2x^3-4x-2$

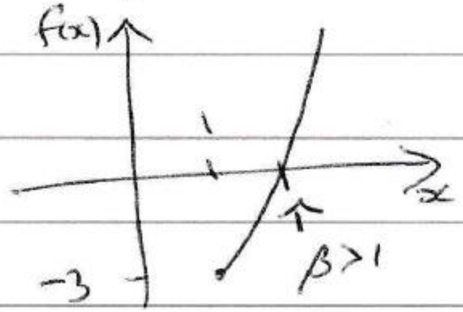
$f(1)=-3$ so for $\beta=1$, $f(\beta)<0$

$f'(x)=4x^3+6x^2-4$

for $x>1$ $f'(x)>0$ so it is increasing

as $x\to\infty$ $f(x)\to\infty$

So $f(\beta)=0$ at some $\beta>1$



The value of β from $a=1$ $b=\beta^2$ and the areas of R and S from these can be used

Let the area of R from this be R^* and the area of S be S^* $(R^* = S^*)$

For a different a where $a>0$ the areas can be transformed with a stretch parallel to the x axis of scale factor a.

This is an affine transformation so the ratio of R to S will remain unchanged

The new value of b will be a times the old value i.e. $b\beta^2$

The STEP specification

- STEP 2 is based on A Level Mathematics and AS Level Further Mathematics.
- The paper has 12 questions across three sections:
  - 8 pure questions
  - 2 mechanics questions
  - 2 probability/statistics questions.

The STEP specification

- STEP 3 is based on A Level Mathematics and A Level Further Mathematics.
- The paper has 12 questions across three sections:
  - 8 pure questions
  - 2 mechanics questions
  - 2 probability/statistics questions.

Problem Solving in the MAT

- You have one STEP III question
- This uses knowledge that is from AS Further Maths
- Read through the question
- Attempt it but think…
    - What's the difference between this and an A level question?
    - What "extra" would my students need to succeed?

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Activity code: 57H86E

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(i) You are given that the transformation represented by A has a line L_1 of invariant points (so that each point on L_1 is transformed to itself). Let (x, y) be a point on L_1.

Show that $\big((a-1)(d-1) - bc\big)xy = 0$.

Show further that $(a-1)(d-1) = bc$.

What can be said about $\mathbf{A}$ if L_1 does not pass through the origin?

(i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$ax + by = x \iff (a-1)x + by = 0 \quad \times (d-1)$

$cx + dy = y \iff cx + (d-1)y = 0 \quad \times b$

$(a-1)(d-1)x + b(d-1)y = 0 \quad (A)$

$bc \; x + b(d-1)y = 0 \quad (B)$

$(A) - (B) \quad \big((a-1)(d-1) - bc\big)x = 0 \iff \big((a-1)(d-1) - bc\big)xy = 0$

From $((a-1)(d-1)-bc)x=0$

either $x=0$ or $(a-1)(d-1)=bc$

If $x=0$ then $by=0$ and $(d-1)y=0$

giving $b=0$ and $d=1$

this also gives $(a-1)(d-1)=bc$

Note the thinking behind this. For a line of invariant points there must exist some points for which $x\neq0$ or $y\neq0$ or x and $y\neq0$

If L_1 does not pass through the origin then it will pass through a point $(k, 0)$ where $k \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix}$$

$ak = k \implies a = 1$ since $k \neq 0$

$ck = 0 \implies c = 0$ since $k \neq 0$

It will also pass trough a point $(0, l), l \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ l \end{pmatrix} = \begin{pmatrix} 0 \\ l \end{pmatrix}$$

$bl = 0 \Rightarrow b = 0$

$dl = l \Rightarrow d = 1$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(ii) By considering the cases $b \neq 0$ and $b = 0$ separately, show that if $(a-1)(d-1) = bc$ then the transformation represented by $\mathbf{A}$ has a line of invariant points. You should identify the line in the different cases that arise.

(iii) You are given instead that the transformation represented by $\mathbf{A}$ has an invariant line L_2 (so that each point on L_2 is transformed to a point on L_2) and that L_2 does not pass through the origin. If L_2 has the form $y = mx + k$, show that $(a-1)(d-1) = bc$.

2019 STEP III Q3

$b \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$ax + by = x \iff (a-1)x + by = 0 \quad \text{(C)}$$

$$cx + dy = y \iff cx + (d-1)y = 0 \quad \text{(D)}$$

For a line of invariant points, these need to be the same straight line

From (C) $\times (d-1)$

$$(a-1)(d-1)x + b(d-1)y = 0$$

but $(a-1)(d-1)=bc$

$$bcx+b(d-1)y=0$$

divide by b since $b \neq 0$

$$cx+(d-1)y=0$$

they are the same straight line and

$(a-1)x+by=0$ $y=\left(\frac{1-a}{b}\right)x$ is its equation

For $b=0$

$$(a-1)(d-1)=bc \quad \Rightarrow \quad (a-1)(d-1)=0$$

either $a=1$ or $d=1$

if $a=1$ $\quad \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$

$$x=x$$

$$cx+dy=y$$

$$cx+(d-1)y=0$$

$y=\left(\dfrac{-c}{d-1}\right)x$ is a line of invariant points

$d \neq 1$

If $d=1$ $\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$ax = x \Rightarrow a = 1$ or $x = 0$

$cx + y = y \Rightarrow cx = 0$ $c = 0$ or $x = 0$

If $a=1$ and $c=0$ it gives $A = I$

Otherwise $x = 0$ is an invariant line of $\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}$

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L_2 is an invariant line of the form $y = mx + k$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx+k \end{pmatrix} = \begin{pmatrix} X \\ mX+k \end{pmatrix}$$

$$(x, y) \longmapsto (X, Y)$$

$ax + b(mx+k) = X \qquad \times m$

$cx + d(mx+k) = mX + k$

$amx + bm(mx+k) = mX \qquad (E)$

$cx + d(mx+k) = mX + k \qquad (F)$

$(F)-(E)$ $\quad (c-am)x + (d-bm)(mx+k) = k$

$(c-am)x + m(d-bm)x + (d-bm)k = k$

$(c-am+dm-bm^2)x + (d-bm)k = k$

True for all x so $\quad c-am+dm-bm^2 = 0 \quad (G)$

and $\quad (d-bm)k = k \quad (H)$

by equating coefficients

From (H) $\quad (d-bm) = 1 \quad$ since $k \neq 0$ (doesn't pass through $(0,0)$)

$\underline{\underline{bm = d-1}}$

From (a) $c + dm = am + bm^2$

$c + dm = m(a + bm)$ $\times b$

$bc + dbm = bm(a + bm)$

but $bm = d - 1$

$bc + d(d-1) = (d-1)(a+d-1)$

$bc = (d-1)(a+d-1) - d(d-1)$

$bc = (d-1)(a+d-1-d)$

$bc = (d-1)(a-1)$

Support for students and teachers

- Next Steps for your A level students
  - next course 10^{th} and 17^{th} March
- On Demand Professional Development
  - MAT and TMUA
  - STEP
- For students:
  - Regular Problem Solving Workshops and one day Problem Solving Conferences– contact your AMSP Area Coordinator
  - Problem Solving Matters – applications opening shortly

About the AMSP

- A government-funded initiative, managed by MEI, providing national support for teachers and students in all state-funded schools and colleges in England.
- It aims to increase participation in AS/A level Mathematics and Further Mathematics, and Core Maths, and improve the teaching of these qualifications.
- Additional support is given to those in priority areas to boost social mobility so that, whatever their gender, background or location, students can choose their best maths pathway post-16, and have access to high quality maths teaching.

Contact the AMSP

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Advanced_Maths