Modelling with differential equations

Tom Button
Avril Steele

Continuing Professional Development Standard

National Centre for Excellence in the Teaching of Mathematics

Aims

- Explore some strategies that can used to help students relate word-based definitions of situations to appropriate differential equations
- Demonstrate how technology can be used to investigate solutions to differential equations

There will be a lot of reading text in this session but that is what we are trying to address!

Differential equations

- Types of differential equation
- Analytical methods for solution
- Modelling

	Content
I1	Find and use an integrating factor to solve differential equations of form $\frac{dy}{dx} + P(x)y = Q(x)$ and recognise when it is appropriate to do so
I2	Find both general and particular solutions to differential equations
I3	Use differential equations in modelling in kinematics and in other contexts
I4	Solve differential equations of form $y'' + ay' + by = 0$ where a and b are constants by using the auxiliary equation
I5	Solve differential equations of form $y'' + ay' + by = f(x)$ where a and b are constants by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $f(x)$ is a polynomial, exponential or trigonometric function)
I6	Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation
I7	Solve the equation for simple harmonic motion $\ddot{x} = -\omega^2 x$ and relate the solution to the motion
I8	Model damped oscillations using 2nd order differential equations and interpret their solutions
I9	Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled 1st order simultaneous equations and be able to solve them, for example predator-prey models

Modelling in kinematics

- Quantities and relationships in kinematics

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} \qquad \left(a = v\frac{\mathrm{d}v}{\mathrm{d}x}\right)$$

- Newton's second law of motion

Resultant force = mass × acceleration $\quad (F = ma)$

First order equation

A tank at a chemical plant has a capacity of 250 litres. The tank contains 100 litres of pure water. Salt water enters the tank at a rate of 3 litres every minute. Each litre of salt water entering the tank contains 1 gram of salt. It is assumed that the salt water mixes instantly with the contents of the tank upon entry.

At the instant when the salt water begins to enter the tank, a valve is opened at the bottom of the tank and the solution in the tank flows out at a rate of 2 litres per minute.

Given that there are S grams of salt in the tank after t minutes, show that the situation can be modelled by the differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3 - \frac{2S}{100 + t}$$

Edexcel Core Pure Mathematics, Paper 1, May 2019, Q5

What changes and why?

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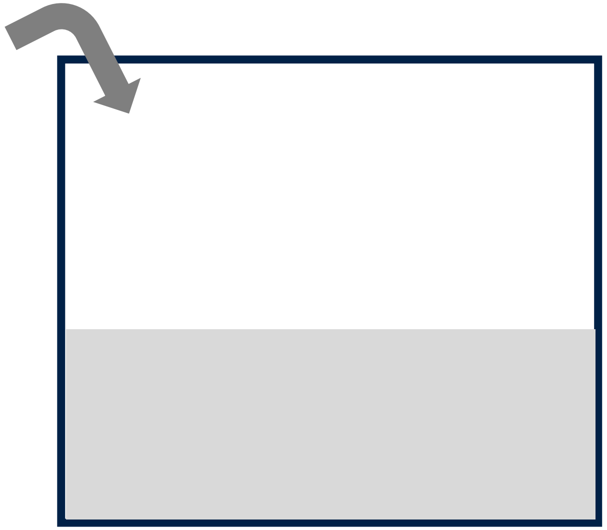
What changes as salt water enters the tank?



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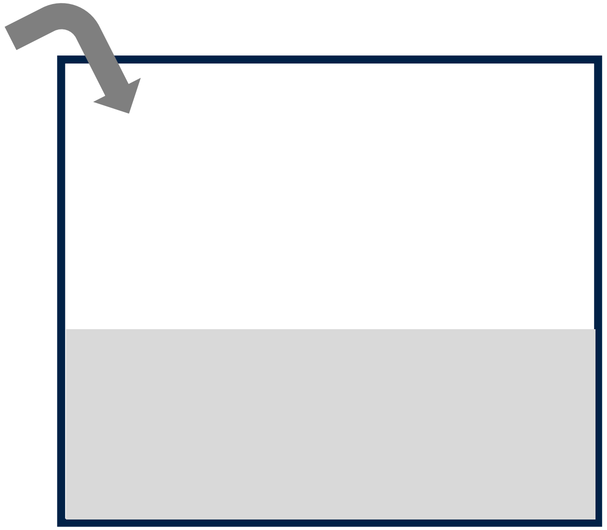

- height of liquid
- volume of liquid
- amount of salt in the tank

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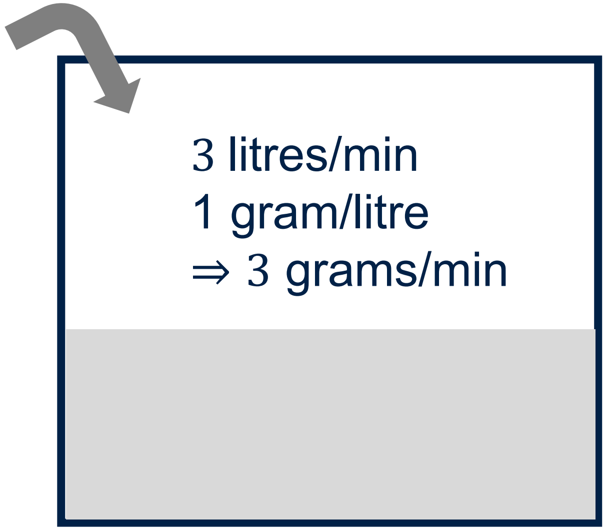

- rate of change of amount of salt

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3$$

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- height of liquid
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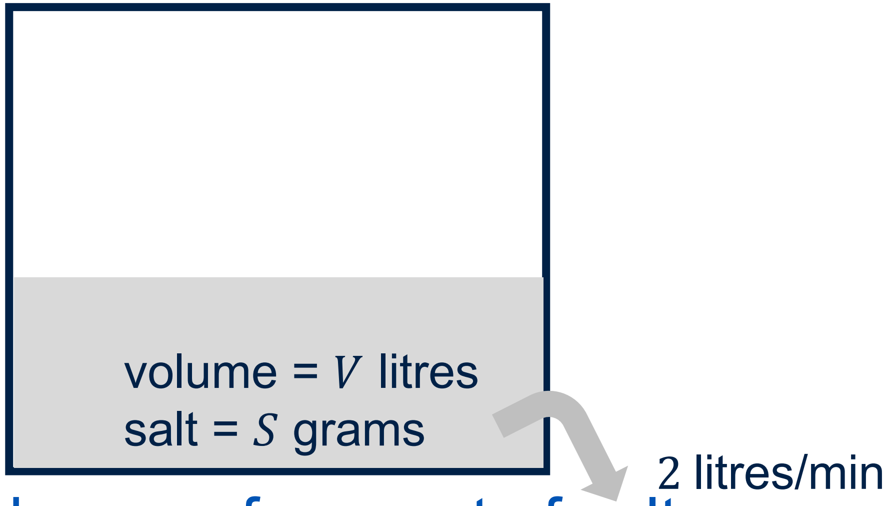

- rate of change of amount of salt

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -2 \times \left(\frac{S}{V}\right)$$

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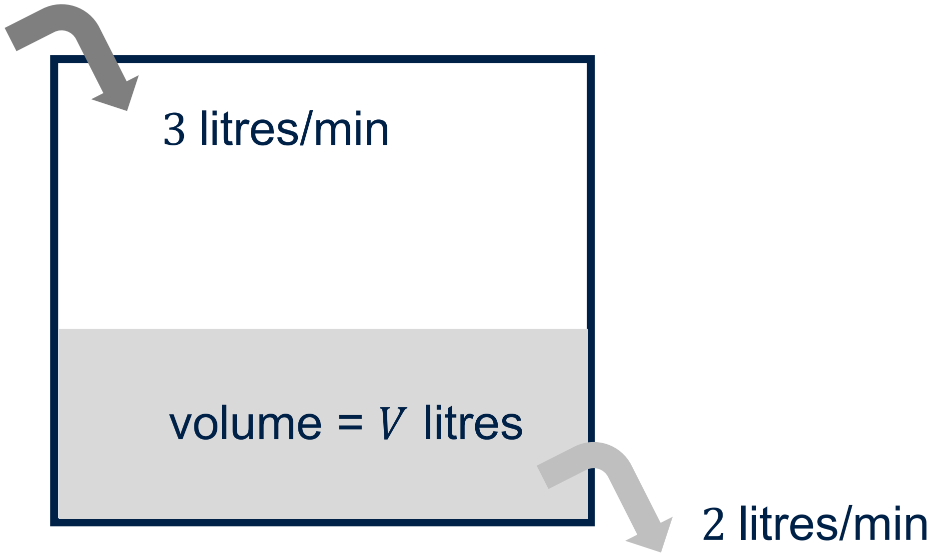
How does the volume change?



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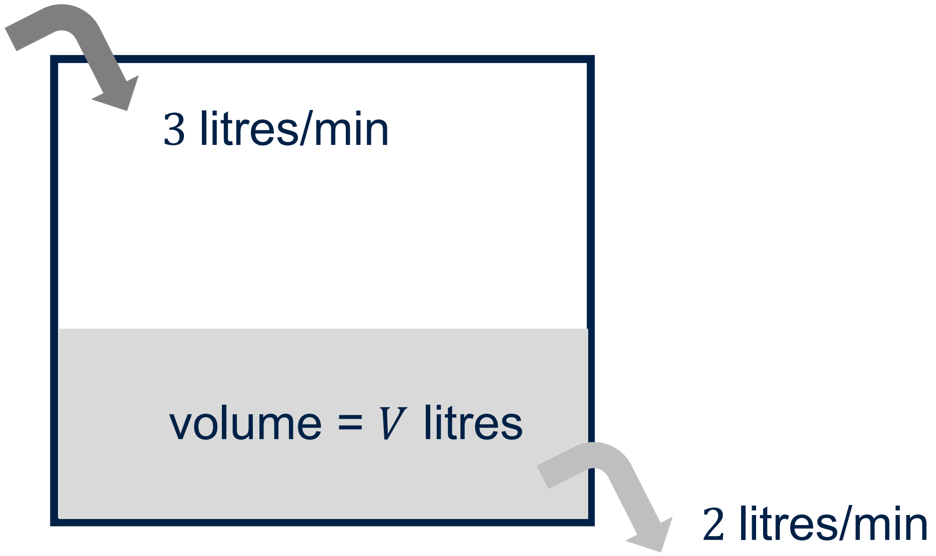

$$V = 100 + (3 - 2)t$$
$$V = 100 + t$$

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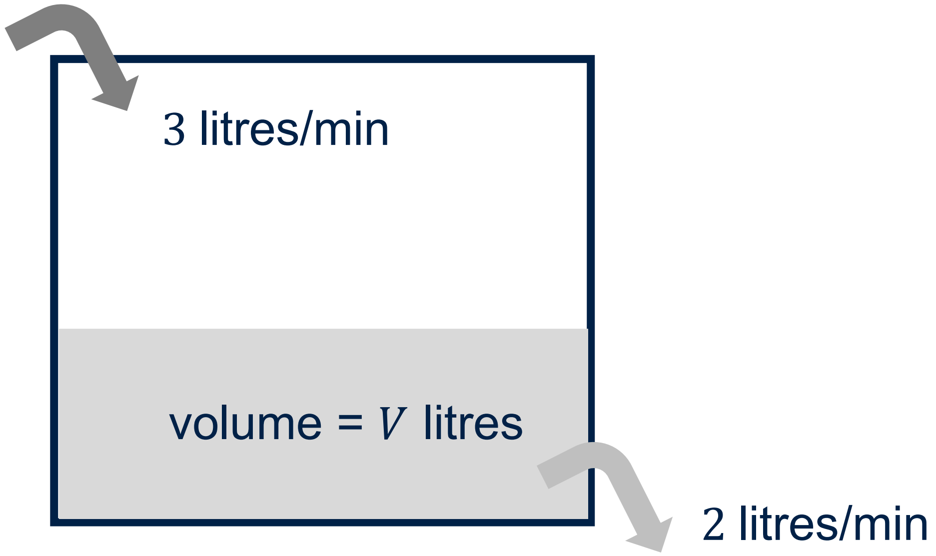

- rate of change of amount of salt

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3 - 2\left(\frac{S}{100 + t}\right)$$

Sketch S against t

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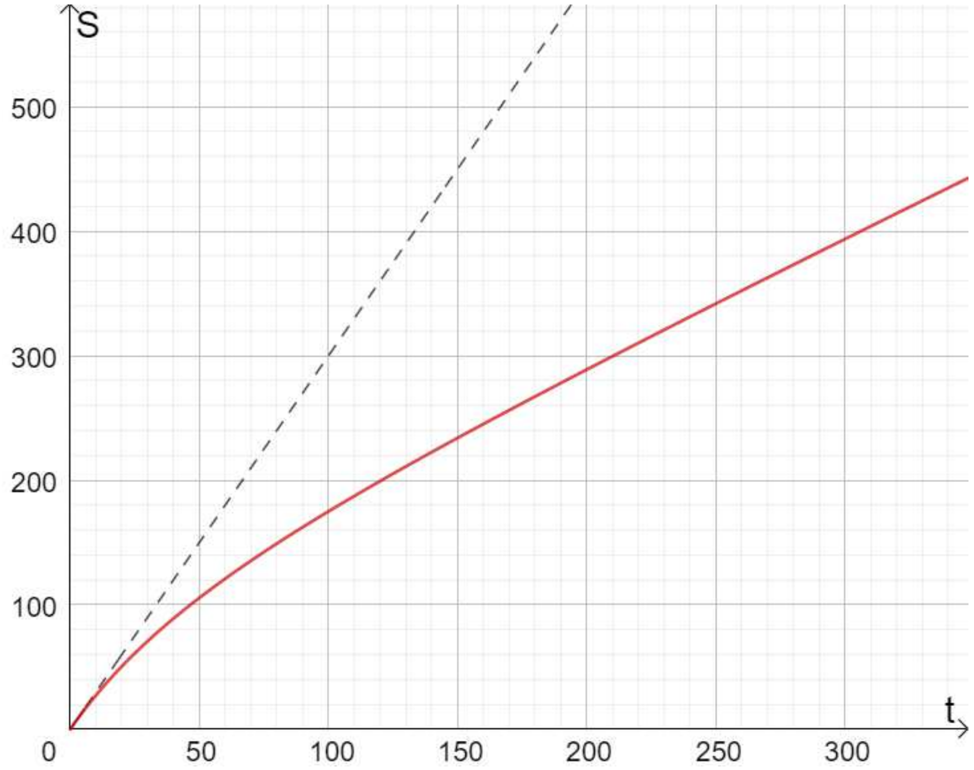

Second order equation

A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, x metres, of the top of the capsule below its initial position at time t seconds is modelled by the differential equation,

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 200\cos t, \qquad t \geq 0$$

where m is the mass of the capsule including its passengers, in thousands of kilograms.

Edexcel SAM, Core Pure Mathematics 1, Q9

A picture of the situation

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Image by Intothewoods29 at English Wikipedia, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=35035786

Some useful questions to explore DEs

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5. When $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2$ what does this mean?

Matching wordy descriptions

Your turn!

Matching wordy descriptions

A.
When a battery is being charged, the rate of increase in charge is proportional to the difference between its current charge and the maximum charge the battery can store.

2.
$$\frac{dy}{dx} = a - by$$

C.
A floating object is displaced vertically slightly and experiences an upthrust proportional to the displacement.

B.
A car is accelerating along a road under the constant force of the engine and against a resistance proportional to its speed.

3.
$$\frac{dy}{dx} = ay(b - y)$$

For an editable version of the Desmos activity used in this session please click here.

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5.
$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} = b$$

C.
A floating object is displaced vertically slightly and experiences an upthrust proportional to the displacement.

4.
$$\frac{d^2y}{dx^2} + ay = 0$$

D.
A model for the population of an insect colony takes into account the fact that as the population grows there are more insects competing for the same food. The rate of growth of the colony is proportional to the product of the size of the population and its difference from a constant maximal population.

3.
$$\frac{dy}{dx} = ay(b - y)$$

E.
The population of a culture of bacteria is growing at a rate proportional to its size.

1.
$$\frac{dy}{dx} = ay$$

Using technology

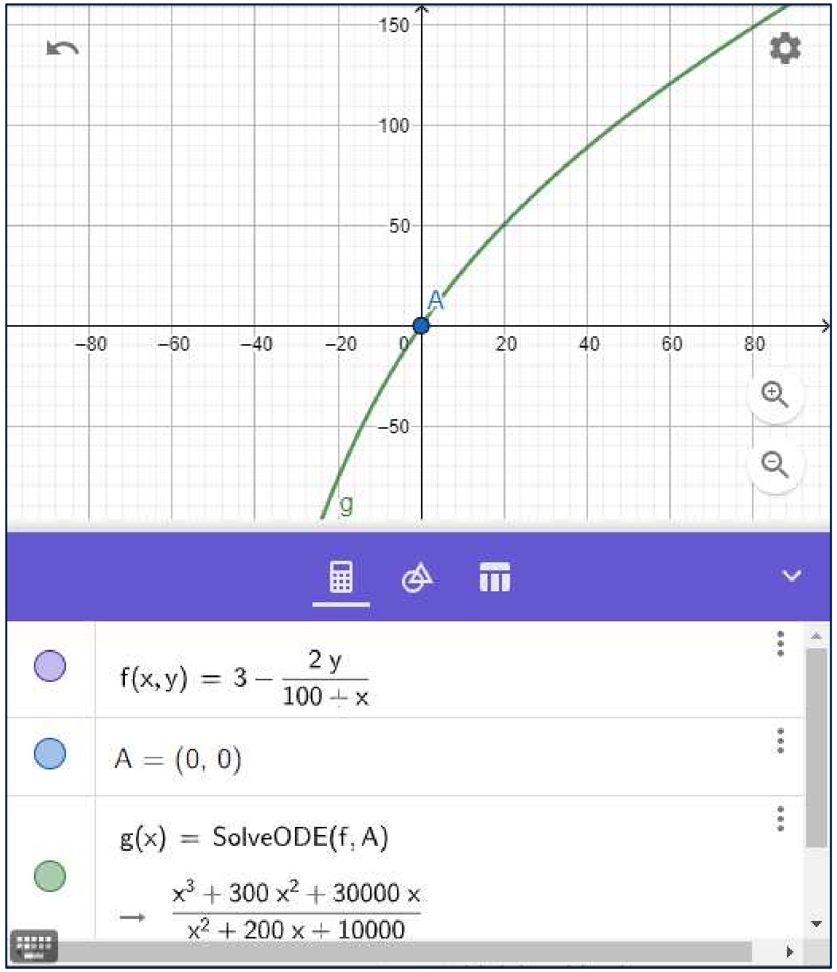
GeoGebra features a **SolveODE** command

e.g. to graph a solution to $\frac{\mathrm{d}s}{\mathrm{d}t} = 3 - \frac{2s}{100+t}$:

Use y for s and x for t.

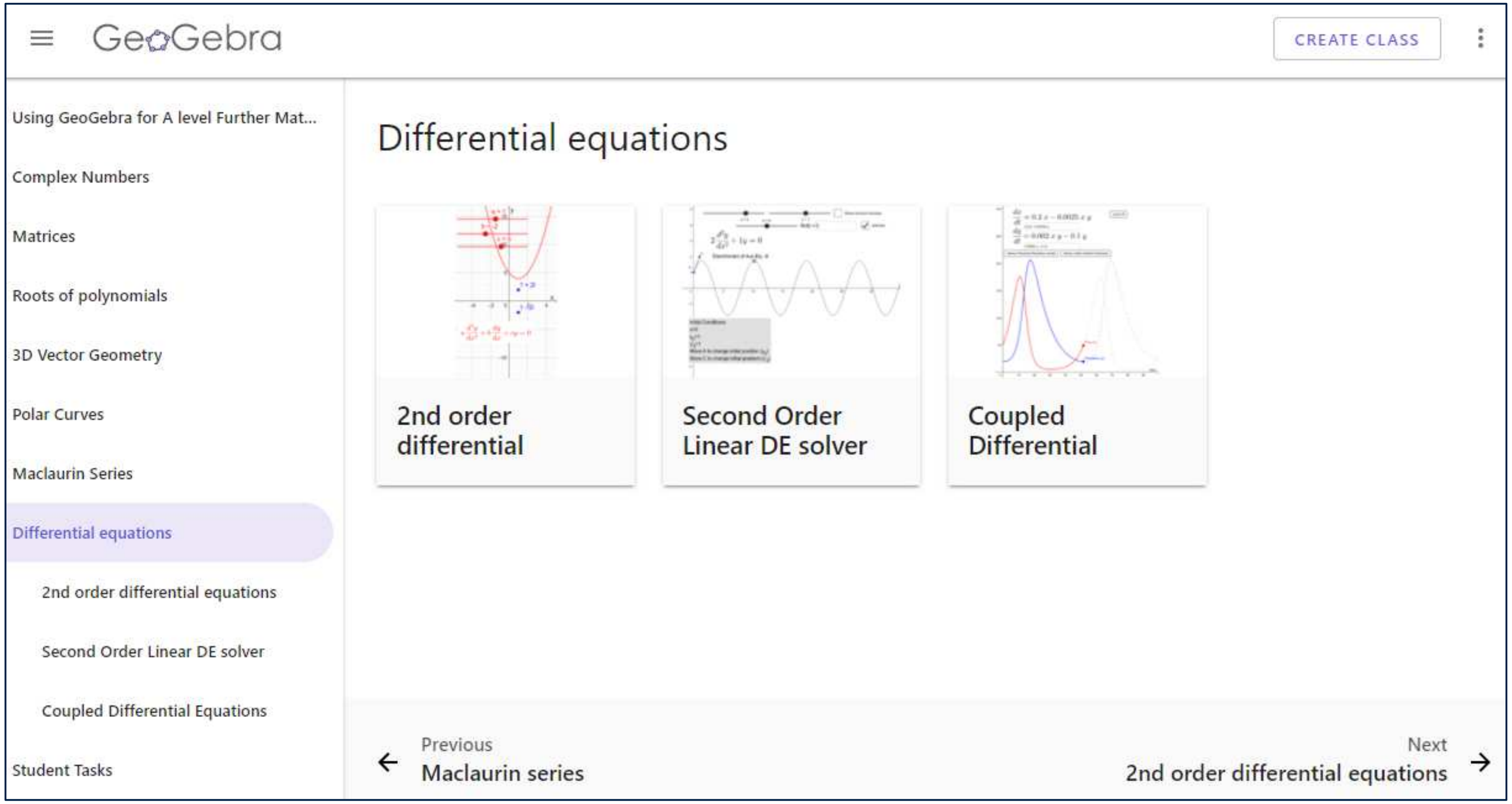
- Enter: **f(x,y) = 3 – 2y/(100+x)**
- Add an initial point: **A=(0,0)**
- To plot the solution: **solveODE(f,A)**

The **SlopeField(f)** command is useful too.



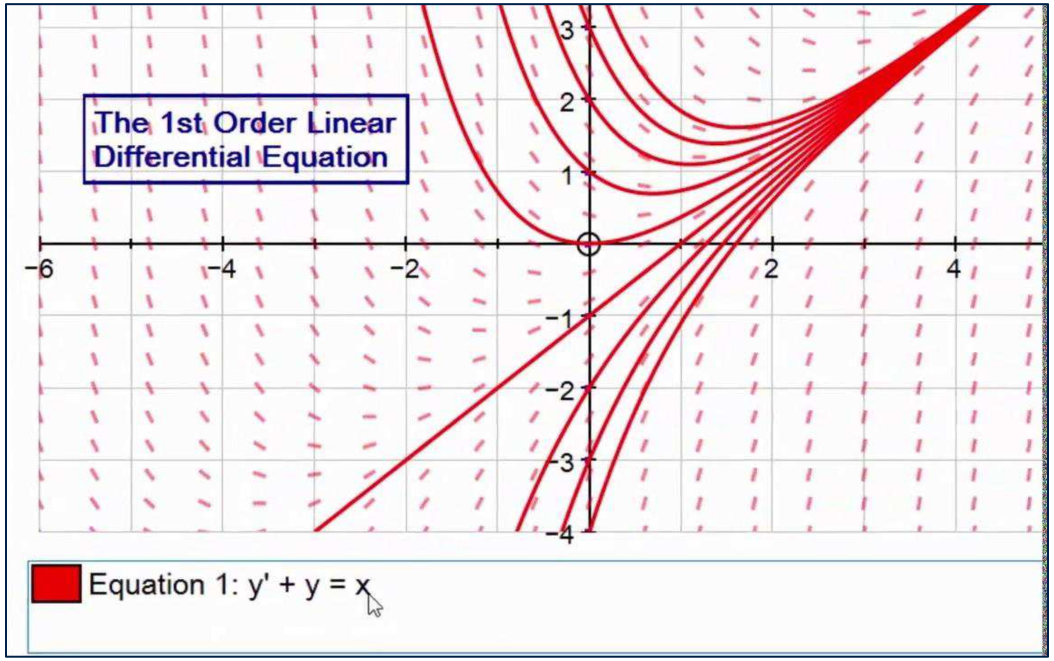
More to explore

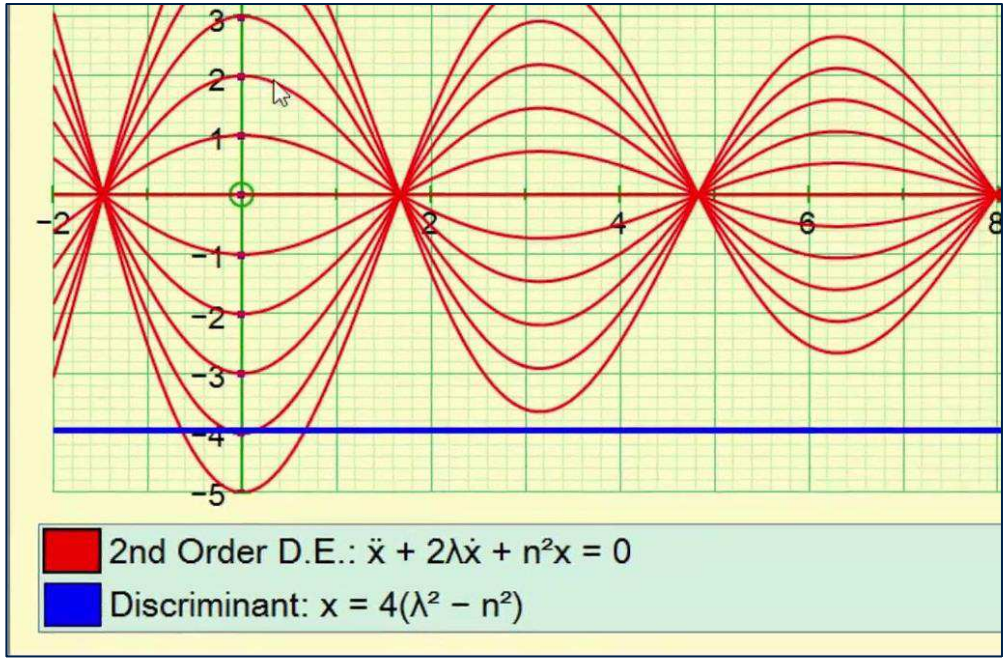
MEI GeoGebra Book for Further Maths: www.geogebra.org/m/XGZP5tbZ



Differential Equations in Autograph

Autograph has some fantastic tools for differential equations.





Thank you!

- Avril Steele: **avril.steele@mei.org.uk**
- Tom Button: **tom.button@mei.org.uk**
  Twitter: **@mathstechnology**

Matching Wordy Descriptions to Differential Equations

Below is a list of differential equations. Which one(s) could be used to model each of the situations A-P on the following pages?

y is the dependent variable, x the independent variable and a, b and c are positive constants.

1) $\frac{dy}{dx} = ay$

2) $\frac{dy}{dx} = -ay$

3) $\frac{dy}{dx} = xy$

4) $\frac{dy}{dx} = a - by$

5) $\frac{dy}{dx} = ay(b - y)$

6) $\frac{dy}{dx} = -a\sqrt{y}$

7) $\frac{dy}{dx} = -ay^2$

8) $\frac{dy}{dx} = a - by^2$

9) $\frac{d^2y}{dx^2} - ax = 0$

10) $\frac{d^2y}{dx^2} + ay = 0$

11) $\frac{d^2y}{dx^2} - ay = 0$

12) $\frac{d^2y}{dx^2} + a\frac{dy}{dx} = b$

It may help students if they appreciate that equations of the forms $\frac{dy}{dx} = a(b - y)$ *and* $\frac{dy}{dx} = -a(y - b)$ *can also be written as equation 4).*

A. The population of a culture of bacteria is growing at a rate proportional to its size.

B. A cup of tea cools at a rate proportional to the difference between the temperature of the tea and the temperature of the surrounding air. (Newton's Law of cooling)

C. In an electrical circuit, the voltage is decreasing at a rate which is proportional to the square of the present voltage.

D. An object is falling through a medium which offers a resistance proportional to its speed.

E. A model for the population of an insect colony takes in to account the fact that as the population grows there are more insects competing for the same food. The rate of growth of the colony is proportional to the product of the size of the population and its difference from a constant maximal population.

F. An upright water barrel is open at the top and water flows out through a hole in the side at the bottom of the barrel. The height of water in a barrel is decreasing at a rate proportional to the square root of the height of the water above the hole.

G. A falling object experiences air resistance proportional to the square of its speed.

H. A floating object is displaced vertically slightly and experiences an upthrust proportional to the displacement.

I. A car is accelerating along a road under the constant force of the engine and against a resistance proportional to its speed.

J. A curve has the property that at any point its gradient is the product of its coordinates.

K. A curve has a gradient which increases with respect to x at a rate proportional to its x coordinate.

L. A curve's gradient increases with respect to x at a rate proportional to its y coordinate.

M. A radioactive substance decays at a rate proportional to the amount of substance present.

N. Air pressure decreases with height above sea level. Under certain conditions the rate of decrease in pressure is proportional to the pressure itself.

O. When a battery is being charged, the rate of increase in charge is proportional to the difference between its current charge and the maximum charge the battery can store.

P. A spring lies on a smooth horizontal table with one end attached to a fixed point. A particle is attached to its free end which is then extended a little. The spring obeys Hooke's Law, that is the tension in the spring is proportional to the extension. The particle is released. What is the equation of motion?

Answers: A-1, B-4, C-7, D-4or12, E-5, F-6, G-8, H-10, I-4or12, J-3, K-9, L-11, M-2, N-2, O-4, P-10