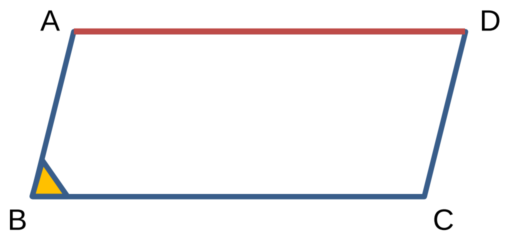
Proof of circle theorems - solutions

There are a number of circle theorems you need to know
Sometimes you can just quote them when you need to give reasons for calculations of angles or lengths
Sometimes you will need to prove them
This exercise will help you see how a proof should be set out

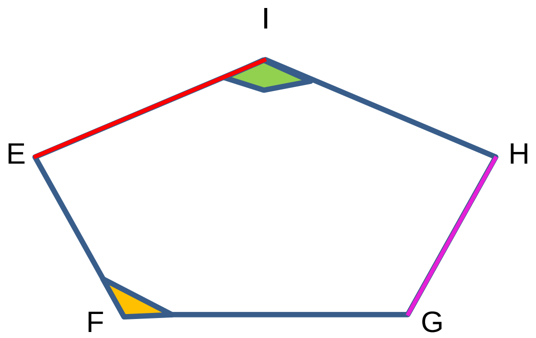
First, make sure you know how to label lines and angles:

Eg.



The red line is called AD or DA

The angle shaded gold is called ABC or CBA



What would you call the red line? EI or IE
What would you call the pink line? HG or GH

What would you call the green angle? EIH or HIE
What would you call the gold angle? EFG or GFE

You also need to know these terms:

Radius

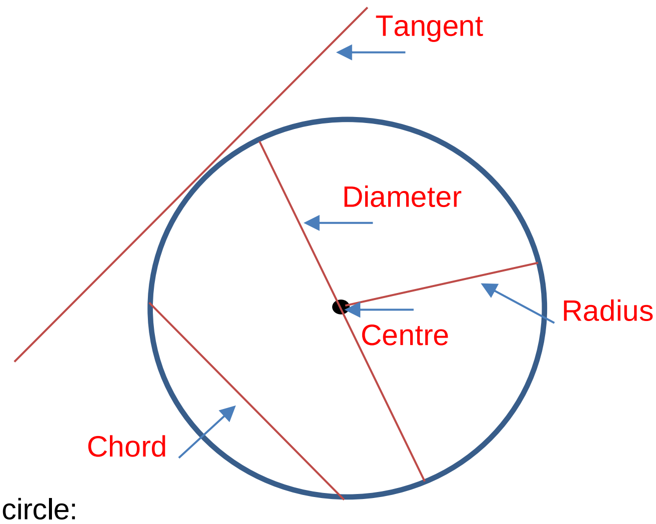
Diameter

Tangent

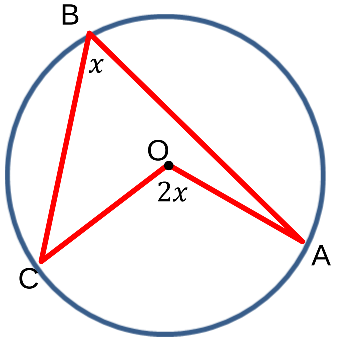
Centre

Chord

Draw and label them on this circle:



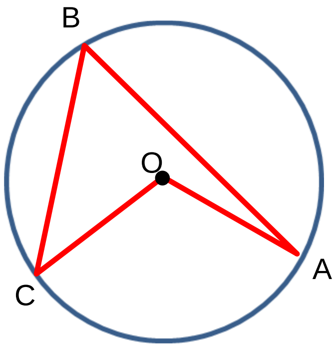
These are the circle theorems you need to know:

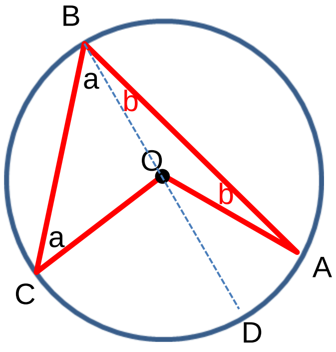


The angle subtended at the centre of a circle is double the angle subtended at the circumference

Angle AOC is double angle ABC

Proof:





Start by drawing a diameter from B, label the other end D

OB and OC are radii, so the triangle COB is isosceles

∴ Angle OBC = OCB (let's call this "a")

∴ Angle BOC = 180 – 2a (because angles in a triangle add up to 180°)

∴ Angle COD = 2a (because angles on a straight line add up to 180°)

So angle COD is 2x angle CBO

In a similar way we can make the same argument for triangle OAB:

OB and OA are radii , so triangle AOB is isosceles

∴ Angle OBA = angle OAB (call this "b")

∴ Angle BOA = 180 – 2b (because angles in a triangle add up to 180°)

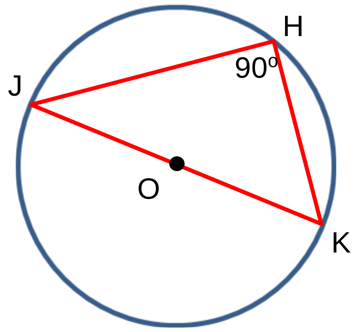
∴ Angle AOD = 2b (because angles on a straight line add up to 180°)

So angle AOD is 2x angle ABO

CBA = a + b

COA = 2a + 2b = 2(a + b) = 2xCBA as required

Note: Once you have proved a theorem, you don't need to prove it again if you need to use it to prove another theorem.

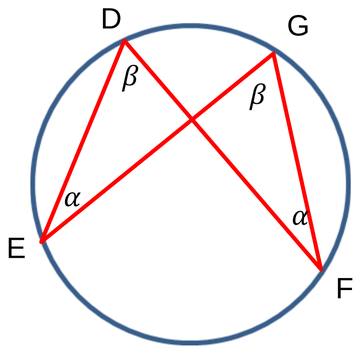


The angle subtended at the circumference by a semi-circle is always 90˚

Proof:

Angle JOK is 180˚ (because the diameter is a straight line)

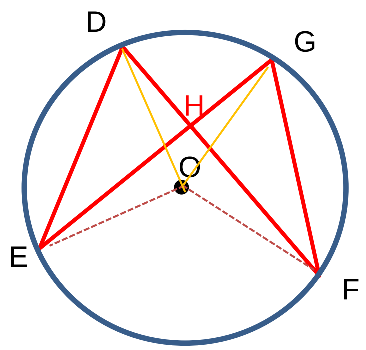
So the angle at the circumference is 180 ÷ 2 = 90° (because the angle at the centre is double the angle at the circumference)



Angles subtended at the circumference in the same segment are equal

EDF = EGF & DEG = DFG

Proof:



Draw in the radii EO and FO

Since the angle at the circumference is half the angle at the centre then

EDF = ½ of EOF

EGF = ½ of EOF

∴ Angle EDF = angle EGF as required.

Now prove that DEG = DFG

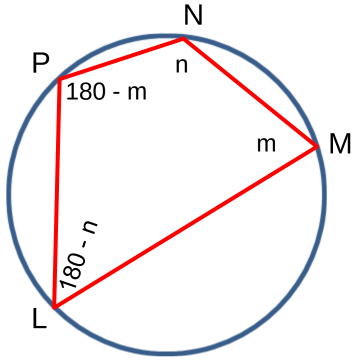
EDF = EGF as proved above

DHE = GHF (because vertically opposite angles are equal)

∴ DEG = DFG (because angles in a triangle always add up to 180°)

Or

Since the angle at the circumference is half the angle at the centre then DEG = ½ of DOG and DFG = ½ of DOG

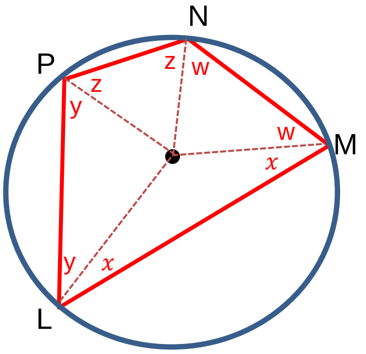


Opposite angles in a cyclic quadrilateral always add up to 180º

PLM+MNP = 180˚

LPN + LMN = 180˚

Proof:



Draw lines from the centre of the circle to each of the vertices of the quadrilateral. Each of these lines is a radius so the quadrilateral has been split into 4 isosceles triangles.

Use a different letter to label the two equal angles in each triangle. (We have used x, y, z & w here but you can use any letter you like).

The internal angles of a quadrilateral add up to 360˚

So $2x$ + 2y + 2w + 2z = 360˚

∴ x + y + w + z = 180˚

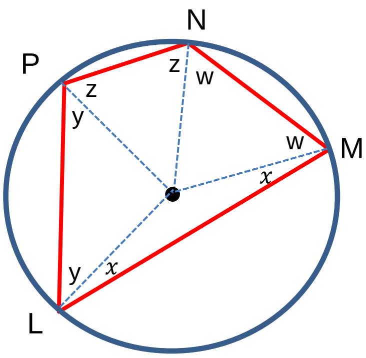
We can pair these angles in any order we like so

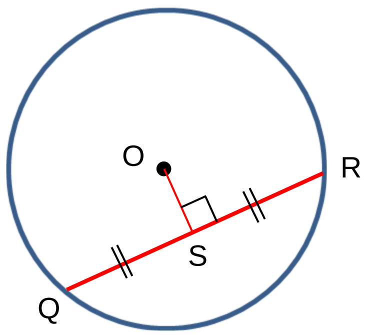
(x + y) + (w + z) = PLM + PNM = 180˚

Or

(x + w) + (y + z) = LMN + LPN = 180˚

pairs of opposite angles add to make 180˚ as required

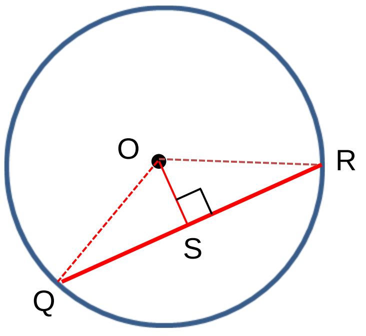




The perpendicular from any chord which passes through the centre will bisect the chord

QS = SR

Proof



Start by drawing radii QO and RO

Angle QSO = 90° because OS is perpendicular to QR and QR is a straight line.

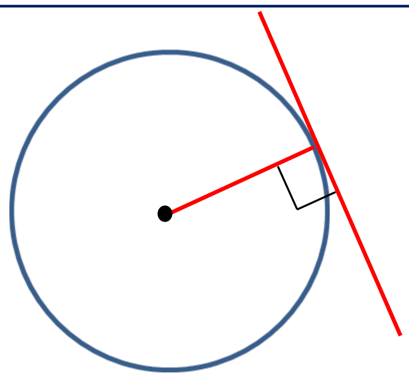
Triangle OQS is congruent to triangle OSR because

QO = OR

And OS is a side on both triangles

And QSO = RSO = 90°

∴ QS = SR as required

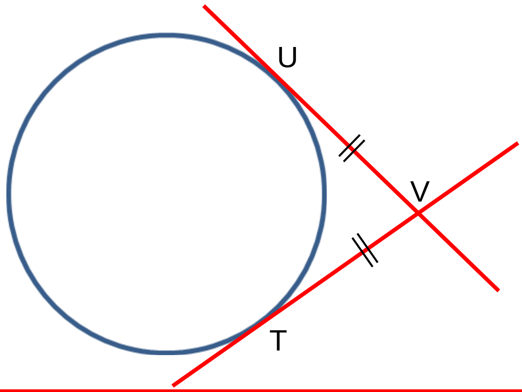


The angle between a radius and tangent which meet at the circumference is always 90°

Proof:

By definition a tangent must be perpendicular to a radius

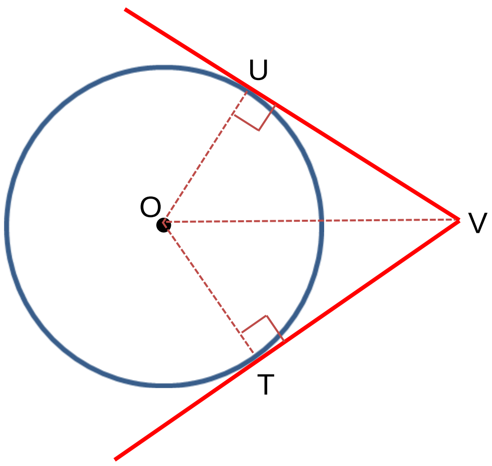
Alternatively you can think of a tangent as a chord that extends beyond the circle, but has zero length inside the circle. Then the line from the centre of the circle (the radius) must be perpendicular to the tangent, as proved in the previous theorem.



Two tangents will always meet at the same distance from the circle.

TV = UV

Proof:



Start by drawing the radii TO and UO

Mark in the right-angles OUV and OTV

Draw in the line OV

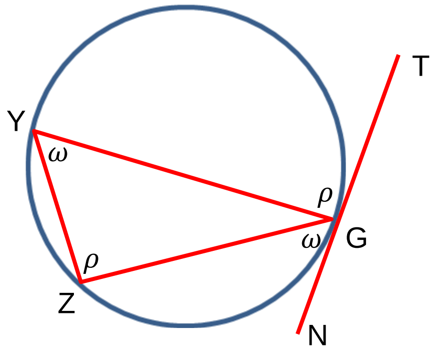
Triangle OUV is congruent to triangle OTV because:

OU = OT

And angle OUV = angle OTV

And OV is a side on both triangles

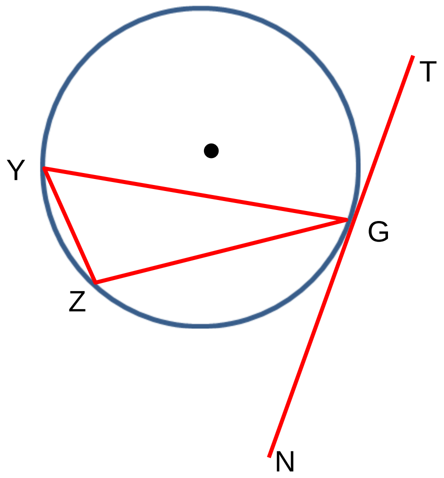
∴ UV = TV as required



The angle between the **tangent** and the side of a circumscribed triangle is **equal** to the opposite internal angle of the triangle.

YGT = YZG & ZGN = ZYG

Proof:



Call angle YGT a and ZGN b, and label these on the diagram.

Now draw a diameter from G, label the other end X
and draw a line from X to Y

XGY = a - 90 (because diameter meets tangent at = 90º)

XYG = 90˚ (because angle at circumference in a semi-circle)

∴ GXY = 180 – 90 – (a-90) = 180-a

∴ YZG = a (because opposite angles in a cyclic quadrilateral add up to 180º)

∴ YZG = YGT as required

Now show that GYZ = b

XGZ = 90 - b (because diameter meets tangent at = 90º)

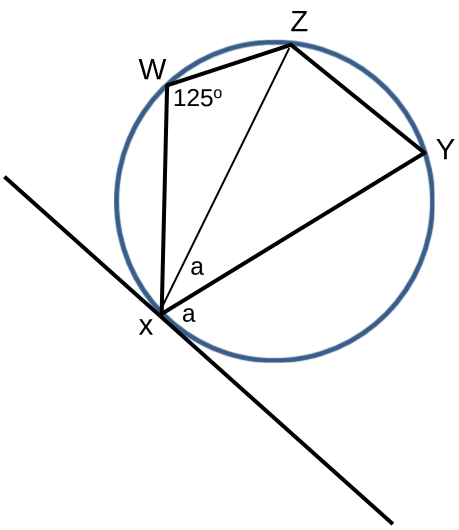
XZG = 90˚ (because angle at circumference in a semi-circle)

∴ GXZ = 180 – 90 – (90-b) = b
∴ GYZ = b (because angles subtended at the circumference in the same segment are equal)

∴ GYZ = ZGN as required

In the GCSE exam you may be asked to work out an angle or a length and give a reason. You can quote any of the circle theorems without proving them first.

For example:



The diagram shows quadrilateral WXYZ inscribed within a circle and a tangent at X.
The angle XWZ is 125°

The diagram is not drawn to scale

a) Write down the size of angle XYZ 55˚

Give a reason for your answer

Opposite angles in a cyclic quadrilateral add up to 180˚

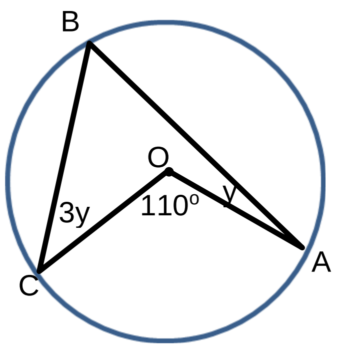
b) Explain how you know that the line XZ is not a diameter

The angle subtended at the circumference in a semicircle is 90˚, angle XWZ is not 90˚ so XZ cannot be a diameter.

c) Work out the size of angle a 62.5˚

Give a reason for your answer
The angle between the tangent and the side of a circumscribed triangle is equal to the opposite internal angle of the triangle, so angle XWZ = angle between XZ and the tangent = 2a = 125˚, so a = 62.5˚

Example 2



Not drawn to scale

a) What is the size of angle ABC? 55˚

Give a reason for your answer
The angle subtended at the centre of a circle is double the angle subtended at the circumference

b) What is the value of y? You must show all your working out
Draw a line AC
OAC is an isosceles triangle with angles 110˚, 35˚ and 35˚.
ABC is a triangle with angles 55˚, (3y+35˚) and (y+35˚)
180=55+ 3y + 35 + y + 35
Y=13.5˚

In questions like this, why do they always say “Not drawn to scale”?
So that students don’t measure the angles but demonstrate that they understand circle theorems and can use them to work them out.