Further Mathematics Support Programme

20 Mathematical Problems suitable for Higher Tier GCSE Students

A collection of 20 mathematical problems to encourage the development of problem-solving skills at KS4.

Each includes suggested questions to ask students to help them to think about the problem and a full worked solution.

The problem sheets are available for free download separately from www.furthermaths.org.uk

CM 12/08/15
Version 1

Problem 1

Can you arrange the numbers 1 – 15 in a triangle where the number below each pair of numbers is the difference between those two numbers?

Here's an example using the numbers 1 – 10:

```
6     1     10    8
   5     9     2
      4     7
         3
```

Can you complete the triangle below?

```
6     ?     ?     ?     ?
   ?     ?     12    ?
      ?     ?     2
         4     ?
            5
```

Suggested Questions to ask students about Problem 1

Note: It's important that students fully understand the term 'difference'. A difference between two numbers is never negative: it's just the distance between the numbers if you were to mark them on the number line. So the difference between 5 and 2 is 3, as is the difference between 2 and 5.

Ask questions like

'What is the difference between 2 and 5?'. Then 'What is the difference between 5 and 2?'.

'If the difference between x and 5 is 2, what could x be? (there are two possibilities)

'If the difference between x and y is 10 what could x and y be?'.

Now relate this to the triangle problem, if necessary asking questions like 'what could the missing numbers be in each of the following diagrams?'.

? 5	11 3	? ?
10	?	2

Getting into Problem 1

Now look at the triangle in the problem. Ask if there are any numbers that can definitely be filled in straight away from the information given.

Once this is done, look at parts of the diagram where there is some information about the missing numbers but not enough to work them out exactly straight away. Ask what the possibilities are and lots of 'what if?' type questions.

Remember that only the numbers from 1 – 15 inclusive can be used and that no number should appear twice. Keep asking question like 'what if this number is a 6?'. Like when doing a Sudoku problem, it might be useful to make rough notes of which numbers could go where before committing to them.

Problem 1 – Solution

1. Here is the initial grid

6 ? ? ? ?

? ? 12 ?

? ? 2

4 ?

5

2. Some numbers can be entered from the bottom of the diagram, 9 then 11 then 1. These three are easy to see:

6 ? ? ? ?

? **1** 12 ?

? **11** 2

4 **9**

5

3. It's worth noting that the numbers in the two positions highlighted below are not immediately obvious based on the difference property of the table alone. There are two possibilities for each of them, if you only consider that property.

6 ? ? ? ?

? 1 12 **14 or 10**

15 or 7 11 2

4 9

5

4. However the position marked 15 or 7 actually has to be 7 because 15 itself cannot be a difference and so it needs to be in the top row. 8 in the second row follows from this.
The position marked 14 or 10 has to be 10 because, of the available numbers, 14 can only be the difference between 15 and 1 but 1 is already used.

6 ? ? ? ?

8 1 12 **10**

7 11 2

4 9

5

5. The remaining numbers to be put in are 3, 13, 14 and 15. With a bit of thought it's possible to see that this is the complete solution.

6 14 15 3 13

8 1 12 10

7 11 2

4 9

5

Problem 2

What number, when multiplied by itself,

is equal to 27 x 147?

Suggested Questions to ask students about Problem 2

It's important that students realise they are looking for the square root of 27 x 147 in this question.

It's worth asking, as an example to make this point,

'What is another name for the number, which when multiplied by itself, equals 57?'.

Ask students to approximate 27 x 147.

Ask whether they can suggest some numbers whose square is this big? They may or may not be able to do this but it's a good thing for them to think about. The idea in this question is that there is a much better way to approach this.

Ask students a good way to find the divisors of a number?

Ask students the following 'here is a number expressed as a product of prime numbers, what is its square root?'

3 x 3 x 3 x 3 x 5 x 5 x 11 x 11 x 17 x 17 x 17 x 17

Getting into problem 2

Remember that you don't have a calculator to do this.

The first thing to think about is the phrase "number, when multiplied by itself, gives…". This means that you're looking for the square root of a certain number.

Here that number is 27 x 147. If you actually do that multiplication by hand you'll get a pretty big number (about 4,000 or so) and it will be difficult to spot the square root. Therefore you are looking for a better way to do this.

This problem is about knowing divisors of 27 x 147 and picking out the one that is the square root.

Finding prime factors is the best way to find out about all the divisors.

An important realisation here is that the prime factors of 27 x 147 are the prime factors of 27 together with the prime factors of 147.

When you write 27 x 147 as a product of prime factors you then need to think about how this tells you about its square root.

Problem 2 – Solution

1. The number which, when multiplied by itself, is equal to 27 x 147 is the square root of 27 x 147. To get an idea of what this might be it's useful to investigate the prime factor of 27 x 147. These will be the prime factors of 27 together with the prime factors of 147.

2. So 27 x 147 = 3 x 3 x 3 x 3 x 7 x 7. Because there are an even number of 3s and an even number of 7s we can write this as a whole number squared:

$$27 \times 147 = (3 \times 3 \times 7) \times (3 \times 3 \times 7) = (3 \times 3 \times 7)^2 = 63^2$$

3. Think about why this would not have been possible if there had not been an even number of each prime factor.

Problem 3

If A is divided by B the result is 2/3.

If B is divided by C the result is 4/7.

What is the result if A is divided by C?

Suggested Questions to ask students about Problem 3

The key to this question is getting students to be able to write down a piece of information like ‘if A is divided by B the answer is 2/3’ as a mathematical equation.

Ask questions like

‘If A is divided by B and the answer is 2/3 which is the bigger A or B?”

‘Give me some examples of number A and B where if A is divided by B the answer is 2/3’

‘Can you write down an equation is satisfied by A and B if A divided by B is 2/3?’

To get students thinking about A divided by C you could ask questions like ‘If A is twice as big as B as B is four times as big as C how many times bigger is A than C’?

Getting into Problem 3

Think about what ‘if A is divided by B the answer is 2/3’ tells you about A and B?

There are lots of different ways to write this. One is

$$\frac{A}{B}=\frac{2}{3}$$

another is that the ratio A : B is 2 : 3. You could even rearrange the above to get 3A = 2B.

Now think about what you need to do. You have a statement about A and B and another statement about B and C but you want to end up with a statement about A and C.

Somehow you need to combine the two pieces of information so that B is no longer involved. What does this remind you of?

Sometimes when you are dealing with more than one equation you try to eliminate some of the letters.

Problem 3 – Solution

1. First of all write down what you know. This is that

$$\frac{A}{B}=\frac{2}{3} \text{ and that } \frac{B}{C}=\frac{4}{7}$$

2. The key thing to realise in this question is that

$$\frac{A}{B}\times\frac{B}{C}=\frac{A}{C}$$

   This is just the usual cancellation law when you multiply fractions.

3. This means that

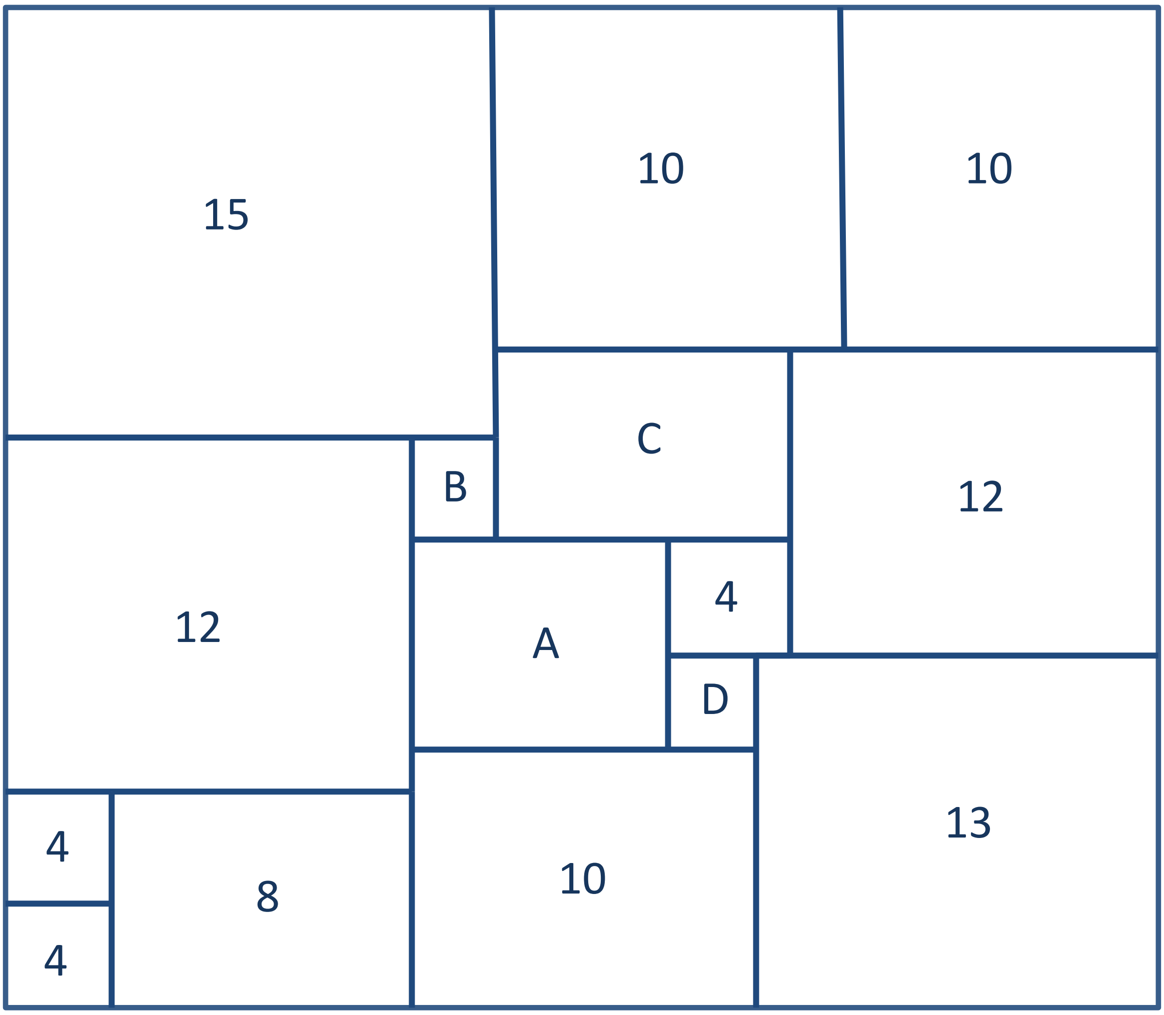
$$\frac{A}{C}=\frac{A}{B}\times\frac{B}{C}=\frac{2}{3}\times\frac{4}{7}=\frac{8}{21}$$

Problem 4

Each shape contained within the largest square is also a square.

The number in each square gives the length of its sides.

What are the values of A, B, C and D?



Suggested Questions to ask students about Problem 4

What is the length of the sides of the big square?

So how long is any horizontal line going from side to side or vertical line going from top to bottom of the big square? (This is a slightly strange question which possibly overcomplicates a rather trivial point so you might not want to use this.)

Write down the equations you get using this fact for some randomly selected horizontal or vertical lines? Are some of the equations easier than others?

Getting into Problem 4

Look at the diagram, think about what you are trying to work out and what information you can get from the diagram.

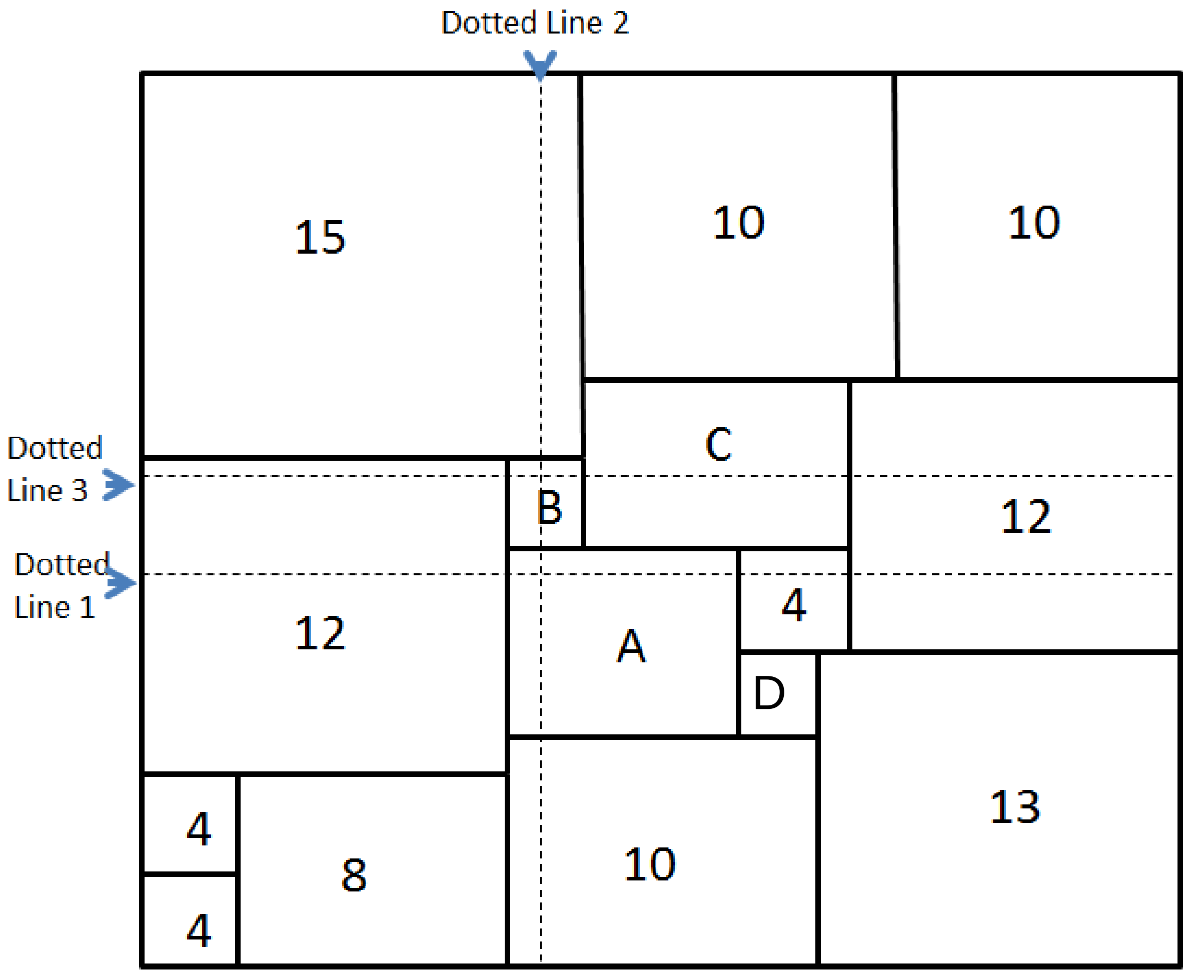
You should be able to work out how long the sides of the big square are.

Now remember that *any* horizontal line going from side to side or vertical line going from top to bottom of the big square will have that length.

Such horizontal or vertical lines will give you different equations involving A, B, C and D.

Can you find any equations this way that just involve one of these letters?

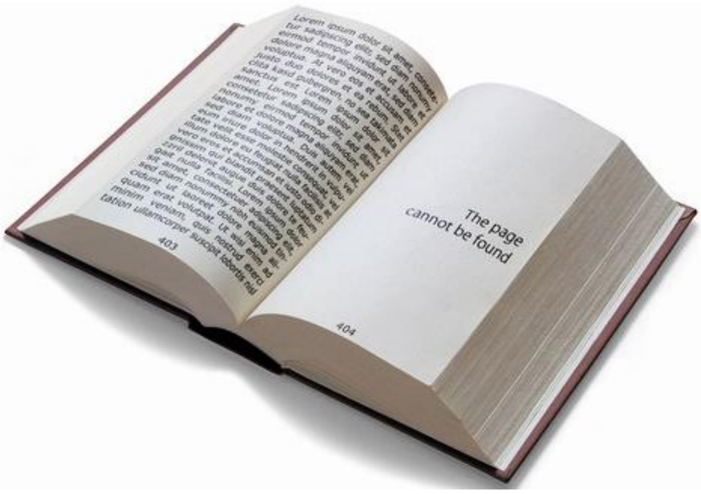
Problem 4 – Solution



1. Looking at the bottom edge of the whole square, it's clear that its side length is 4 + 8 + 10 + 13 = 35.
2. Now, by looking at Dotted Line 1, it can be seen that 12 + A + 4 + 12 = 35. Solving this equation gives A = 7.
3. By looking at Dotted Line 2, and remembering that A = 7 it can be seen that 15 + B + 7 + 10 = 35. This means that B = 3.
4. By looking at Dotted Line 3 and remembering that B = 3, it can be seen that 12 + 3 + C + 12 = 35 which means that C = 8.
5. Finally, by using either the value of A or the value of C we can get a value of D = 3.

Problem 5

Over the course of numbering every page in a book, a mechanical stamp printed 2,929 individual digits.



How many pages does the book have?

Suggested Questions to ask students about Problem 5

If a book had nine pages how many printed digits would there be for the page numbers?

If a book has ninety-nine pages how many printed digits would there be?

How about one hundred and ninety-nine pages?

Using this, have a guess as to how many pages this book might have? Check whether you answer is correct?

If it's not correct do you need to increase or decrease the number of pages? How can you work out how many you need to increase it or decrease it by?

Getting into Problem 5

To do this problem you need to start thinking about the number of digits on page numbers of a book starting from the start.

The first nine pages are pages 1, 2, 3, 4, 5, 6, 7, 8, 9. The total number of digits here is just 9.

After that you get pages which have two digits in them. These are pages 10 to 99 inclusive. How many pages like this are there (be careful)? What is the total number of digits that appear on these pages. So how many digits would there be on the first 99 pages? If the book in the problem has more than this number of digits then it must have more than 99 pages.

Keep going using this approach.

Once you think you have a rough idea of how many pages there are you will need to think carefully about how to get the exact number of pages.

Problem 5 – Solution

1. Our book needed 2,929 digits to be stamped. Start by getting a feel for how many pages this might be. To do this it's good to think about the pages in sequence, from the beginning, thinking about how many digits there are in each page number.

2. Pages 1-9 have one digit each, so the total number of digits to be stamped for these pages is 9. This means our book has more than 9 pages.

3. Pages 10-99 all need two digits to be stamped. From pages 10 to 99 inclusive there are 90 pages. So, for these 90 pages 90 x 2 = 180 digits need to be stamped. This means that from page 1 – 99, 189 digits need to be stamped. So our book has more than 99 pages.

4. Pages 100-999 all need three digits to be stamped. There are 900 pages from page 100 to page 999 inclusive and so 900 x 3 = 2700 digits need to be stamped. This means that from page 1 – 999, 2889 digits need to be stamped. So our book has more than 999 pages.

5. Our book has 2,929 digits stamped. How many more than 2,889 is this? This is just 2,929 – 2,889 = 40.

6. Pages 1000, 1001, 1002 now all have 4 digits. This must mean that there are 40/4 = 10 more pages after page 999.

7. This means our book has 1009 pages.

Problem 6

Can you make the two columns of numbers add up to the same total by swapping just two cards between the columns?

1	3
2	4
7	5
9	8

Suggested Questions to ask students about Problem 6

What do the two columns add up to at the moment?

So how many bigger does the smaller one need to get (or equivalently how many smaller does the bigger one need to get)?

Think about possible swaps. If you swap the 2 in the left hand column with the 5 in the right hand column how much will the left hand column go up by and how much will the right hand column go down by? So would that swap make the two columns equal?

What about other swaps? Is there any pattern in what happens with the swaps?

Getting into Problem 6

Remember that when you swap a number in the left hand column with a number in the right hand column, both totals will change.

Think about examples of this. If you swap the 2 in the left hand column with the 5 in the right hand column, the left hand column total will go up by 3 and the left hand column total will go down by 3.

You may now start to spot that this is going to be tricky.

You might be able to prove that any swap you try will not do what you need.

You might be able to find another reason that you can't do this. Ask yourself what the implications would be if you *were* able to do this?

It's useful to think about odd and even numbers to really see what is going on in this problem.

Problem 6 – Solution

1. This is a tough question. You may have noticed that whatever you seem to try you can't get the two columns to add up to the same total.

2. In fact it's impossible and here is why:

   Suppose you could make the two columns add up to the same number, call this n. That would mean that the total on all the cards was n + n = 2n.

   Now n is a whole number and so 2n is an even number.

   But if you add up all the cards you get 1 + 2 + 3 + 4 + 5 + 7 + 8 + 9 = 39, not an even number.

3. This is actually an example of 'proof by contradiction'. The thing we did wrong, which led to the contradiction in the above is to suppose that we could make the columns add to the same number. Therefore it must be **impossible** to make the columns add to the same number.

Problem 7

What is the smallest number divisible by

1, 2, 3, 4, 5, 6, 7, 8 and 9?

Suggested Questions to ask students about Problem 7

Ask students whether the smallest number divisible by both 6 and 10 is 6 x 10 = 60?

What is the smallest number divisible by both 6 and 10? How do prime factorisations help with this?

So will the answer to this problem be 1 x 2 x 3 x 4 x 5 x 6 x 7 x 8 x 9?

How can prime factorisations help with this?

Getting into Problem 7

Think about how you would do this problem if there are only two numbers.

It might be useful to think about the smallest number that is divisible by both 6 and 10.

Take the prime factorisations of 6 and 10. These are 6 = 2 x 3 and 10 = 2 x 5.

The smallest number that is divisible by both 6 and 10 must have a prime factorisation which includes all of these primes. It needs to include each prime the maximum number of times it appears in any of the factorisations of 6 and 10.

So the smallest number divisible by 6 and 10 is 2 x 3 x 5 = 30.

Problem 7 – Solution

1. To do this it's useful to think of the prime factorisations of all the numbers from 2 to 9. These are listed below.

Number	Prime factorisation	Primes in the prime factorisation, with repeats
2	2	2
3	3	3
4	2 x 2	2, 2
5	5	5
6	2 x 3	2, 3
7	7	7
8	2 x 2 x 2	2, 2, 2
9	3 x 3	3, 3

2. Now think about the prime factorisation of the smallest number which is a multiple of all the numbers from 1 to 9. It needs to include all the primes which are listed in the right hand column above. The number of times we need each prime is given by the highest number of occurences of it in the right hand column above. The primes appearing are 2, 3, 5, 7. The highest number of occurences of 2 is 3 (in 8 = 2 x 2 x 2). If each prime appears that many times we will have the smallest number that is divisible by all of the above.
3. This gives us the number 2 x 2 x 2 x 3 x 3 x 5 x 7 = 2520. You can see from the prime factorisation that it is divisible by all the numbers from 1 to 9.

Number	Appearance as a divisor
2	**2** x 2 x 2 x 3 x 3 x 5 x 7
3	2 x 2 x 2 x **3** x 3 x 5 x 7
4	**2 x 2** x 2 x 3 x 3 x 5 x 7
5	2 x 2 x 2 x 3 x 3 x **5** x 7
6	2 x 2 x **2 x 3** x 3 x 5 x 7
7	2 x 2 x 2 x 3 x 3 x 5 x **7**
8	**2 x 2 x 2** x 3 x 3 x 5 x 7
9	2 x 2 x 2 x **3 x 3** x 5 x 7

Problem 8

This problem is about dividing with a remainder.

As an example of this, when 29 is divided by 6 the answer is 4 and the remainder is 5.

When a whole number x is divided by 5, the answer is y and the remainder is 4.

When x is divided by 4, the answer is v and the remainder is 1.

Can you write an equation expressing y in terms v?

Suggested Questions to ask students about Problem 8

Ask students how they can write a statement like “When x is divided by 5 the answer is y and the remainder is z” as a mathematical expression.

Ask students how the fact that 29 = 7 x 4 + 1 shows that when 29 is divided by 4, the answer is 7 and the remainder is 1.

Ask them about the significance of 1 < 4 in the example above (remainders must be less than the number you are dividing by).

Getting into Problem 8

Think about an example like ‘When 29 is divided by 3 the answer is 9 and the remainder is 2’.

Another way of saying that ‘When 29 is divided by 9 the answer is 3 and the remainder is 2’ is to state that 29 = 3 x 9 + 2. ← Remainder

↑ Answer

This idea should give you two algebraic expressions and you can go on from there.

Problem 8 – Solution

1. You need to find a way of writing these ideas using algebra
2. If 29 divided by 4 is 6 with remainder 5, we can write this as:

$$29 = (4 \times 6) + 5$$

Similarly, if x divided by 5 is y with remainder 4, we can write this as:

$$x = (5 \times y) + 4$$
$$x = 5y + 4$$

And finally, if x divided by 4 is v with remainder 1, we can write this as:

$$x = (4 \times v) + 1$$
$$x = 4v + 1$$

3. Now we have 2 formulae for x, which we can equate. This gives:

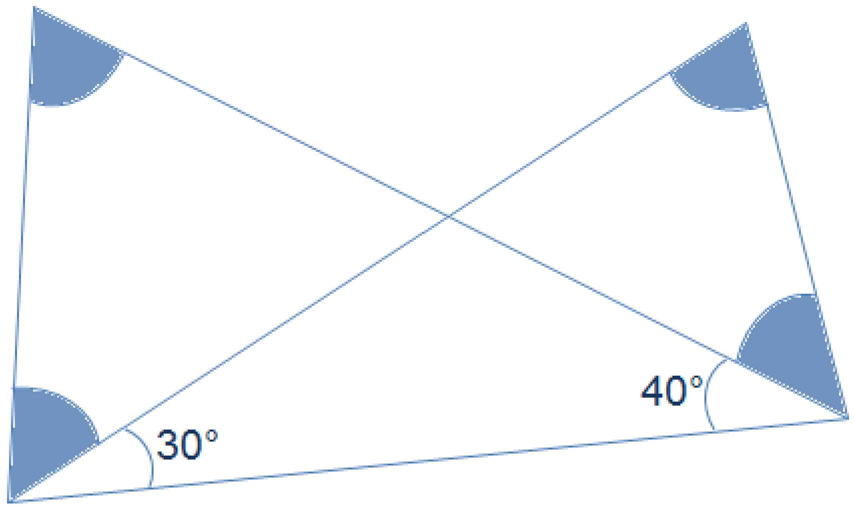
$$5y + 4 = 4v + 1$$

Now rearrange this to make y the subject

$$5y = 4v – 3$$
$$y = 4/5v - 3/5$$

Problem 9

In the diagram below what is the sum of the four shaded angles?



Suggested Questions to ask students about Problem 9

Ask students about how they can find a way to talk about the individual angles in the diagram without having to point at them. (You are trying to steer them towards labelling them here.)

Ask students what facts about angles they can remember and what might be relevant to this problem.

Once the angles are labelled ask the students to form any equations at all involving the label sizes.

Ask the students whether their equations will enable them to work out the individual angle sizes? Do they need to be able to work out the individual angle sizes to answer the question?

Getting into Problem 9

To do a problem like this it's often useful to give all the unknown quantities (in this case the angles) letter names. You might label the four missing angles a, b, c, and d.

Now think about ways that you can get information about a, b, c and d.

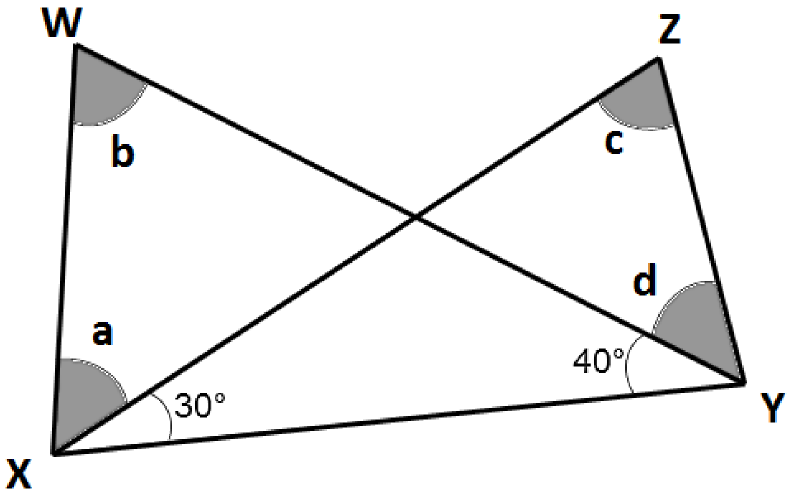
You know that the angles in a triangle add up to 180 so look for triangles containing a, b, c and d.

Each time you find one, write down the equation which says that its angles add up to 180.

Now look at what you have got. Remember that you don't need to find a, b, c, and d individually, just a + b + c + d.

Problem 9 – Solution

1. Look at triangle WXY first.



Its angles should total 180˚, so we know that…

$$b + (a + 30) + 40 = 180$$

Rearranging

$$a + b = 180 - 40 - 30$$
$$a + b = 110^\circ$$

2. Now let's examine triangle XYZ. Its angles should also total 180˚, so we know that…

$$30 + (40 + d) + c = 180$$

Rearranging

$$c + d = 180 - 30 - 40$$
$$c + d = 110^\circ$$

3. Combining this, we see that the sum of the 4 angles is…

$$\begin{aligned} a + b + c + d &= (a + b) + (c + d) \\ &= 110 + 110 \\ &= 220^\circ \end{aligned}$$

Problem 10

Five numbers are arranged in order from least to greatest:

$$x, x^3, x^4, x^2, x^0$$

Where does $-x^{-1}$ belong in the list above?

Suggested Questions to ask students about Problem 10

One of the numbers in the list given doesn't depend on x, which one? What does this tell you about x?

When you square a number which is between 0 and 1 is the answer bigger or smaller than the number you started with? Combined with the information you've been given what does this tell you about x?

What happens if you cube a number between -1 and 0? Is the answer to the right or left of the original number on the number line?

What happens if you cube a number which is less than -1? Is the answer to the left or to the right on the number line?

Getting into Problem 10

To do this question aim to try to work out whether x is positive or negative and what its magnitude is compared to 1 by thinking about the answers to the questions above.

To answer these questions, it's useful to consider examples in each case.

Once you know this you should be able to work out where -1/x should go.

Problem 10 – Solution

We are told that

$$x < x^3 < x^4 < x^2 < x^0$$

What does this tell us about x?

First of all $x^0 = 1$ so we know that $x < 1$.

If $x = 0$ then $x^4 = x^2$ which is not the case.

If $0 < x < 1$ then we would have $x^2 < x$ (think about $x = 0.5$ for example). So it must be the case that $x < 0$.

If $x < -1$ then we would have $x^3 < x$ (then about $x = -2$ for example).

If $x = -1$ then $x^3 = x$ which is not the case.

From all the above we can deduce that $-1 < x < 0$.

Now think about $-x^{-1}$. If you take a number between -1 and 0 then the reciprocal is less than -1. So the negative of the reciprocal is greater than 1.

Problem 11

Explain why the 1st of March is always on the same day of the week as the 1st of November

Make a deduction about the day of the week that 31st May falls on compared to the 1st August.

Suggested questions to ask students about Problem 11

Ask questions like

What do you need to think about when you are doing the first part of this question?

How could you get caught out?

How does what you have found in the first part of the question help with the second part?

Getting into Problem 11

First of all work out how many days there are between 1st March and 1st November. It’s important to define what is meant by ‘between’, which dates are counted and which are not. One this is done then a division should show the reason why the two dates are on the same day of the week.

Think about 31st May and 1st August, it should be possible able to use the same techniques to see the relationship between the days on which these two dates fall.

Problem 11 – Solution

1. Let's look at the cycle of days through the week.

   Take any start date. We know that 7 days later, it will be the same day of the week as the start date. In addition, 14 days, 21 days, 28 days etc. later will also be the same day of the week.

2. So let's count the number of days between 1st March and 1st November in each year. We add up 31 days for March, 30 for April, 31 for May, 30 for June, 31 for July, 31 for August, 30 for September, and 31 for October. This gives 245 in total.

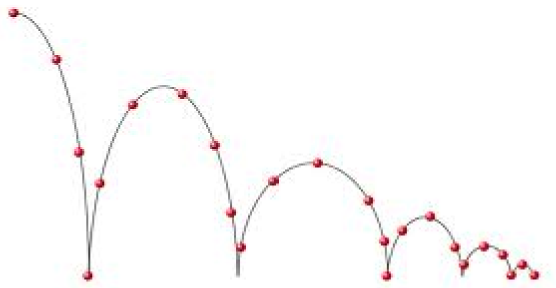
3. Now let's look at 31st May and 1st August.
   There are 1 + 30 + 31 = 62 days between these dates, which is not a multiple of 7.

   However, 62 is equal to 8 lots of 7 with a remainder of 6. So we can say for sure that if 1st August will be the day of the week previous to whatever day of the week the 31st May was.

   For example, in 2011, 31st May was a Tuesday, and 1st August was the day of the week before Tuesday – a Monday!

Problem 12

A ball is dropped and bounces up to a height that is 75% of the height from which it was dropped.



It then bounces again to a height that is 75% of the previous height and so on.

How many bounces does it make before it bounces up to less than 20% of the original height from which it was dropped?

Suggested questions to ask students about Problem 12

The key to this question is getting students to be able to write down a sequence of the heights reached on sequential bounces. It could be approached by writing down terms in the sequence or programming a spreadsheet to calculate the heights for various release heights

Look carefully at the diagram, what information can you get from this?

How would you label the diagram to make it more helpful to you?

Is it easier to start with an example where you decide what height the ball is dropped from?

On the first bounce the ball reaches 75% of this height it was dropped from, how could you write this in a simple way?

What about the next bounce? Can you write this in terms of the height the ball was originally dropped from?

Getting into Problem 12

Firstly look at the diagram and think about the information that can be obtained from it as it is. Then think about the extra information that might be added to it to make it even more helpful. It is probably helpful to give a letter, say h, to the height the ball is dropped from.

It might help to start by choosing a specific height to drop the ball from and working out the answer in this case, then you can try to generalise

On the first bounce the ball reaches 75% of this height it was released from it would be useful to write 75% as a decimal so that you can write the height of the first bounce as a decimal multiplied by the release height. Now do this for the second bounce multiplying the two decimals. Keep doing this for each bounce until the decimal multiplying h is equivalent to a percentage less than 20%.

Problem 12 – Solution

1. Let's call the original height the ball was dropped from h.
   After the first bounce we know it reached 75% of this height, so we can write this as 0.75h.

   After the next bounce it reached a height of:

   $$0.75 \times 0.75h = 0.5625h$$

2. The question is asking to find the number of bounces it took for the height of the ball to be less than 20% of the original height. In other words less than 0.2h.

3. After 3 bounces, the ball reached 0.75 x 0.5625h = 0.4219h. This is only 42.19% of the original height, so it is not small enough yet.

4. After 4 bounces, the ball reached 0.75 x 0.4219h = 0.3164h. This is still not small enough.

5. After 5 bounces, the ball reached 0.75 x 0.3164h = 0.2373h. Still not small enough!

6. Finally, after 6 bounces, the ball's height was 0.75 x 0.2373h = 0.1780h, or 17.8% of its original height, which is less than 20%! So the answer is 6 bounces.

Problem 13

Adam has three coins in his pocket, and they are all different from each other. Ben has three coins in his pocket and they are all the same as each other.

Adam has half as much money as Ben.

What are the coins they each have?

What happens if Adam has twice as much money as Ben?

Suggested questions to ask students about Problem 13

Ask ‘how might you start a problem like this?’.

Get students to ask themselves lots of ‘what if’ questions about Adam and Ben’s totals. For example ‘What if Adam had a 5p, a 10p and a 20p? What would Adam’s total be? What would Ben’s total need to be?

If this doesn’t result in the students coming up with a strategy, ask students whose total (Adam or Ben) it’s easiest to think about?

If this doesn’t provide a way in, be more specific e.g. ‘what if Ben’s coins are all 20p? What would Adam’s total need to be?’

Getting into Problem 13

A first reaction might be to try to do this with algebra considering an equation like $2(x + y + z) = 3a$. Is this sensible (think about the number of variables in the equation)?

It’s important to realise that this question is only possible because of the finite number of possibilities for x, y, z and a. This suggests an approach based on considering possibilities.

Since Ben has only one type of coin it’s much easier to think about the total of his coins than the total of Adam’s coins (there are fewer possibilities). It’s easy to list the possible totals for Ben.

Halving each one gives the possible totals for Adam. Which ones are achievable with three distinct coins?

In the second question Adam has twice as much money as Ben it should be possible to repeat the logic used in the first part of the question to see whether there is a solution.

Problem 13 – Solution

1. Let's look at the possible types of coins that Ben could have in his pocket, and the total amount of money this would be.
2. We can also use this to see how much Adam would have in each case, because we know it is half as much as Ben.
3. Then we can ask ourselves whether it's possible to make Adam's amount with three different coins.

Ben	Adam	Make with 3 different coins?
3 x 1p	1.5p	NO – can't have 0.5p!
3 x 2p	3p	NO – the smallest 3 coins make 8p!
3 x 5p	7.5p	NO – can't have 0.5p!
3 x 10p	15p	NO
3 x 20p	30p	NO
3 x 50p	75p	YES
3 x £1	£1.50	NO
3 x £2	£3.00	NO

4. The only way it would work is if Ben had three 50p coins in his pocket, totalling £1.50. This would mean that Adam had 75p in total, made up of a 50p coin, a 20p coin, and a 5p coin.

5. If Adam has twice as much money as Ben then

Ben	Adam	Make with 3 different coins?
3 x 1p	6p	NO
3 x 2p	12p	NO (1 x 10p and 2 x 1p)
3 x 5p	30p	NO (1 x 20p and 2 x 5p)
3 x 10p	60p	NO (1 x 50p and 2 x 5p)
3 x 20p	£1.20	NO (1 x £1 and 2 x 20p)
3 x 50p	£3.00	NO (1 x £2 and 2 x 50p or 3 x £1)
3 x £1	£6.00	NO
3 x £2	£12.00	NO

In this case there is no solution if all the coins must be different but there would be several solutions if two coins could be the same.

Problem 14

p and q are two numbers each greater than zero.

$$\sqrt{(p^2 + 5q)} = 8$$

$$\sqrt{(p^2 - 3q)} = 6$$

Find the values of p and q.

Suggested questions to ask students about Problem 14

If the students have already done simultaneous equations it's worth asking if the equations remind them of any problems they have come across before. If they haven't studied simultaneous equations they can still reason their way through this problem.

Try asking students to identify what is difficult about these two equations.
Once they have done this ask how they might make the problem look a bit simpler.

Is it easier to find p first or q first? Does it matter which you do first?

How many answers are there? How many of these answer the question correctly?

Getting into Problem 14

Students may not have come across simultaneous equations with squares and roots in before so it's important to understand that you can't you just subtract one from the other straight away.

Remember that if you square the LHS of each equation, you must also square the RHS.

Once this is done, there are different approaches that can be used to solve the equations, but it is a good idea to consider whether it easier to find p first or q first. Does it matter which you do first?

Finally think about how many viable answers are there to the problem then re-read the question. How many of these answer the question correctly?

Problem 14 – Solution

To solve this problem we need to manipulate some algebra. We are starting with 2 equations:

$\sqrt{(p^2 + 5q)} = 8$

$\sqrt{(p^2 - 3q)} = 6$

First of all, let's square both sides of each equation to get rid of the square root…

$p^2 + 5q = 64$ (1)

$p^2 - 3q = 36$ (2)

We can solve this pair of simultaneous equations by taking equation (2) away from equation (1)

$p^2 – p^2 + 5q – (-3q) = 64 – 36$

$8q = 28$

$q = \frac{28}{8}$

Now we have to substitute our value of q into one of the equations to get p; it doesn't matter which one we use

Using equation (1) gives

$p^2 + 5 \times \frac{28}{8} = 64$

$p^2 + \frac{140}{8} = 64$

$2p^2 + 35 = 128$

$2p^2 = 93$

$p^2 = \frac{93}{2}$ So $p = {}^{+}\sqrt{\frac{93}{2}}$ or ${}^{-}\sqrt{\frac{93}{2}}$, but we know $p > 0$, so $p = {}^{+}\sqrt{\frac{93}{2}}$

Problem 15

A set of five numbers has:

a mode of 24

a median of 21

a mean of 20

What could the numbers be?

Suggested questions to ask students about Problem 15

Students will probably already have a good understanding of the terms: mode, median and mean. Questions to ask could be

Think of any group of five numbers with a mode of 24. What does this fact alone tell you about the number of 24s in the list?

Think of any group of five numbers with a median of 21. It's probably best to think of the numbers in ascending or descending order. What exactly does this fact tell you if you were to write the numbers like this?

How you would calculate the mean of a group of five numbers? If the mean is 20 what must the total be?

Getting into Problem 15

If the mode is 24, then 24 must appear more times than the other numbers in the set. What is the best number of 24's to include?

If the median number is 21, then 21 must be in the middle of the set,
What does this tell you about the position of the two 24s if the list is written in order?

The mean of the numbers must be 20 so use this fact to work out the total of the five numbers and some possible values for the two other numbers in the list.

Think about whether there could be more than one solution to this problem and , if there is, how many soutions are there?

Problem 15 – Solution

1. If the mode is 24, we need 24 to appear more times than the other numbers in the set. Let's have two 24's in our set, and the other three numbers different to each other.
2. If the median number is 21, we need to have 21 in our set, and also two numbers lower, and two numbers higher. Our two 24's are higher, so we now just need two different numbers lower than 21.
3. The mean of the numbers must be 20. Let's call the last two unknown numbers a and b. To have a mean of 20, we need:

$$\frac{a + b + 21 + 24 + 24}{5} = 20$$

$$\Rightarrow a + b + 21 + 24 + 24 = 100$$

$$\Rightarrow a + b + 69 = 100$$

$$\Rightarrow a + b = 31$$

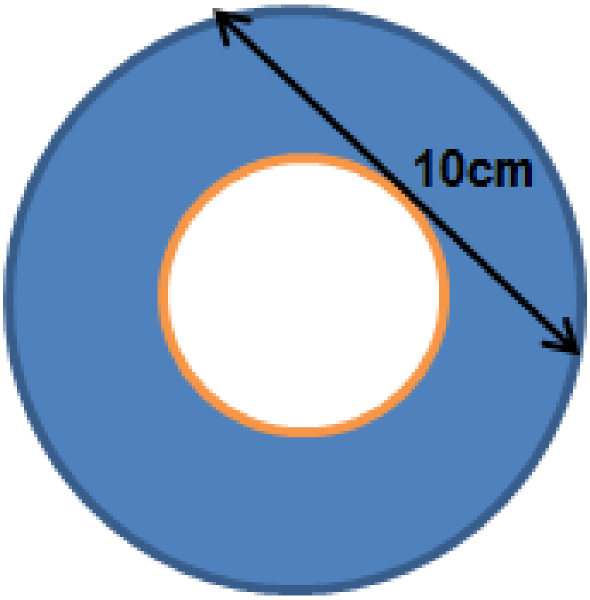
4. We can now pick a and b to be any two numbers that add up to 31, e.g. a = 17 and b = 14 would give the solution

14, 17, 21, 24 and 24

But there are several other correct solutions.

Problem 16

Can you work out the shaded area in the diagram (the line shown just touches the smaller circle)?



Suggested questions to ask students about Problem 16

At first glance it looks very difficult but it isn't as hard as it looks though it does involve bringing together several pieces of maths. Try to get students to decide on the really important pieces of information that they need to think about by asking questions such as

What does the blue area depend on?

Can you add labels to the diagram that will help you to decide on an expression for the blue area?

Can you write the expression for the blue area using formulae that you know and then simplify it?

Can you use the 10cm line to draw a triangle in the circle with vertices at the centre of circle, one end of the 10cm line) and the midpoint of the line?

Which theorem should you use to do calculations involving right angled triangles?

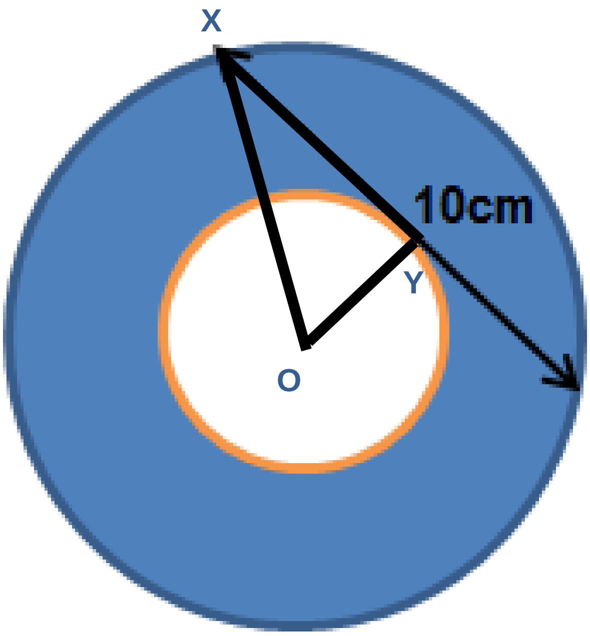
Getting into Problem 16

The blue area will depend on the radii of the two circles so start by giving both of the radii letter names; call the radius of the small circle *r* and the radius of the big circle *R*.
Now write an expression for the shaded area in terms of the radii of the two circles.

Given that you are looking to find the difference between the squares of two lengths, this might suggest using Pythagoras' theorem. R would need to be the length of the hypotenuse and r the length of one of the other sides, so hunt for that right angled triangle!

Problem 16 – Solution

1. Start by calling the radius of the small circle *r* and the radius of the big circle *R*. The shaded area will have an area of
   $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$



2. Now, using the 10cm line given, draw a triangle in the circle with vertices O (centre of circle), X (one end of the 10cm line) and Y (midpoint of the line).

3. We now have a right angled triangle, with sides
   XY = 5cm, OX = *R* and OY = *r*.
   (It is right angled because a radius of a circle passing through the midpoint of a chord will create a right angle.)

4. Using Pythagoras' theorem,

   $R^2 = 5^2 + r^2$
   $R^2 - r^2 = 25$

5. Substitute this into our formula for the shaded area $\pi(R^2 - r^2)$ that we got in step 1 gives

$$A = \pi(R^2 - r^2) = 25\pi$$

Problem 17

You have eight coins that all look identical but only one is solid gold.

The solid gold coin weighs slightly more than the fakes.

You can use the balance only twice. How can you work out which is the real gold coin?

Suggested questions to ask students about Problem 17

The key to this problem is that if you put some coins on the scales and it balances, the heavier coin is not one of the ones on the scale.

Ask students to think about all possible things that could happen on the second weighing if, for the first weighing they

- put four on each side
- put three on each side
- put two on each side
- put one on each side

Getting into Problem 17

Now you need to think about what the first weigh will be.
You can only use the balance twice so after the first weighing there is only one more weigh so you need to think about what will happen next for all the different things you could do on the first weighing.
It's helpful to split the coins into groups, but how many groups, two, three, four, and how many coins in each group?

Listing the possibilities and all their outcomes is a sensible approach to this problem.

Problem 17 – Solution

1. First of all, split the 8 coins into three groups, two groups with 3 coins in, and one group of the remaining 2 coins.

2. Place each of the two groups of 3 coins on each side of the weighing scales. There are 2 possible outcomes

   (i) If it *balances* then all of the coins must be same, so none can be the gold one. If this happens take the 2 coins that were not weighed, and place one on each side of the scales. The heavier coin is the gold one!

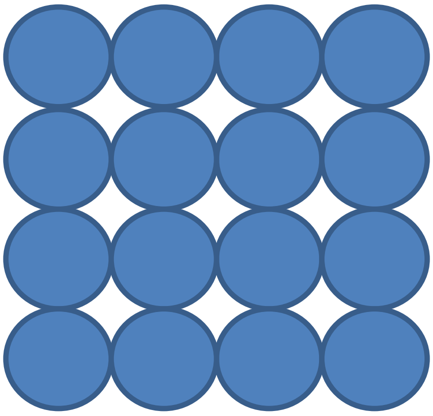
   (ii) If it *doesn't balance* then one group is heavier and must contain the gold coin. Take this group and choose any 2 coins from it. Place one on each side of the scales. Again there are 2 possibilities

   (a) If it *balances* then these two coins are identical and cannot be gold so the gold coin must be the coin that wasn't weighed this time.

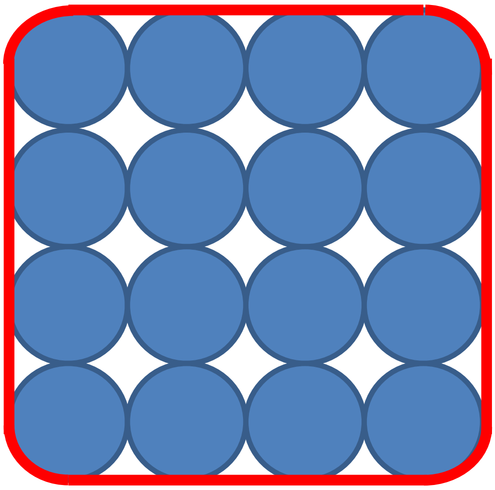
   (b) If it *doesn't balance* then one coin is heavier, and this is the gold coin!

Problem 18

Sixteen pipes, each with a diameter of 10cm, are fastened tightly together with a metal band. How long is the band?



Suggested questions to ask students about Problem 18



First of all make sure students understand the that the band goes round the edge of the pipes and fits as tightly as possible.

Suggest that students draw a diagram showing where the band will go.

Can you divide the band into pieces so that you can calculate the length of each piece?

What formulae will you need to use for each part of the calculation?

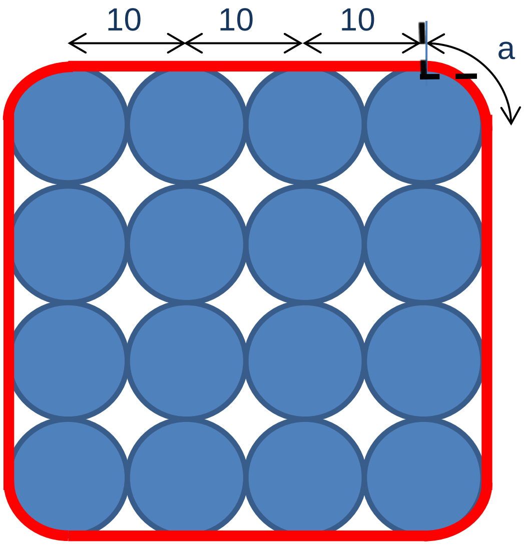
Getting into Problem 18

Start by breaking down the band into sections as this will help with calculating its length

Use the diameter of the pipes to work out the length of the straight edges of the band.

Each corner of the band is part of a circle, what fraction of a circle is it?

Problem 18 – Solution



1. Start by breaking down the ribbon into sections as shown in the diagram. This will help us to calculate its length

2. Each of the straight edges of the band has length 30cm as it is 3 pipe diameters.

   There are four sides, so
   4 x 30cm = 120cm.

3. Each corner is part of a circle. The length, represented by a in the diagram, will be equal to one quarter of a pipe's circumference.
   This is $\frac{\pi d}{4} = \frac{10\pi}{4}$ cm
   There are four corners, so in total the length of the corner sections is
   $4 \times \frac{10\pi}{4} = 10\pi$ cm

Adding everything together, the length of the metal band is
$120 + 10\pi$ cm

Problem 19

Find the sum of any three consecutive numbers.

What do you notice about the total?

Is this true for any three consecutive numbers?

Can you **prove** why this is true?

Suggested questions to ask students about Problem 19

Before starting, check that students understand what ‘consecutive’ means.

What is a good place to start with this problem?

How many sets of numbers do you need to try?

Is there a fixed number of trials you should do?

Can you spot a pattern? Do you need to try more numbers to check your pattern?

Does this prove that your idea is correct?

What is the difference between showing that something works for some numbers and proving that it will work for all numbers? What techniques do you need to use?

If n is a number, how do you write down the next number in terms of n? What about the number before n?

Getting into Problem 19

The best way to start this problem is by trying it with some numbers. Start with small numbers then try some bigger ones and see whether there is a pattern.

Start by choosing any three consecutive numbers e.g. 5, 6 and 7.

Try it with several other sets of consecutive numbers until you think you have spotted a pattern.

To prove that this will always work you need to use algebra so you will need to introduce and define some letters to stand for the three consecutive numbers, what’s the best way to do this?

By adding the three consecutive terms, you should be able to see that the pattern will always work and, because this has been done algebraically, the letters could stand for any numbers so you have proved that it is true for all numbers.

Problem 19 – Solution

1. Start by choosing any three consecutive numbers e.g. 5, 6 and 7. Their total is 18 which is equal to 3 multiplied by 6. 6 is the middle number.

2. This could just be coincidence so try another example, 12, 13 and 14.
   Their total is 39, which again equals 3 times the middle number.

3. To prove that this will always work you need to use algebra

   Call the middle number n, so our three consecutive numbers will be one less than n, n and one more than n: $n - 1, n, n + 1$

   Adding these together gives $(n - 1) + n + (n + 1) = 3n - 1 + 1 = 3n$ which is 3 multiplied by the middle number.

   This proves that, for any three consecutive integers, the total will always be 3 times the middle number.

Problem 20

Choose any three consecutive numbers.

Multiply the first number by the third number.

Square the second number.

What do you notice?

Is this true for any three consecutive numbers?

Can you **prove** why this is true?

Suggested questions to ask students about Problem 20

The approach to this problem is very similar to that for question 19. Before starting, check that students understand what ‘consecutive’ means.

What is a good place to start with this problem?

How many sets of numbers do you need to try?

Is there a fixed number of trials you should do?

Can you spot a pattern? Do you need to try more numbers to check your pattern?

Does this prove that your idea is correct?

What is the difference between showing that something works for some numbers and proving that it will work for all numbers? What techniques do you need to use?

If n is a number, how do you write down the next number in terms of n? What about the number before n?

You will end up with two brackets that each have two terms in, can you remember how to multiply these together?

Getting into Problem 20

The best way to start this problem is by trying it with some numbers. Start with small numbers then try some bigger ones and see whether there is a pattern.

Start by choosing any three consecutive numbers e.g. 5, 6 and 7.

Try it with several other sets of consecutive numbers until you think you have spotted a pattern.

To prove that this will always work you need to use algebra so you will need to introduce and define some letters to stand for the three consecutive numbers, what’s the best way to do this?

Be careful when multiplying the first and third numbers - you need to multiply two brackets.

By adding the three consecutive terms, you should be able to see that the pattern will always work and, because this has been done algebraically, the letters could stand for any numbers so you have proved that it is true for all numbers.

Problem 20 – Solution

1. Start by choosing any three consecutive numbers e.g. 2, 3 and 4.
   Multiply the first number by the third number 2 x 4 = 8
   Square the second number: $3^2 = 9$.

   9 is one greater than 8.

2. Try another example to see whether this is repeated e.g. take the numbers 10, 11 and 12.
   Multiply the first number by the third number 10 x 12 = 120
   Square the second number: $11^2 = 121$

   Again 121 is 1 greater than 120.

Now try to prove algebraically that this is always the case

Call the middle number n, so our three consecutive numbers will be
One less than n, n and one more than n: $n - 1, n, n + 1$

Multiply the first number by the third number
$(n - 1)(n + 1) = n^2 - n + n - 1$
$= n^2 - 1$

Square the second number: n^2

So the second number squared will always be one more than the first and third multiplied together.