


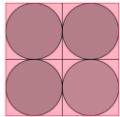

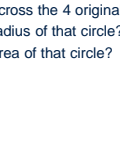

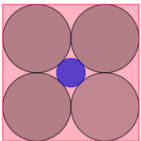


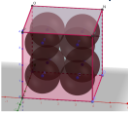







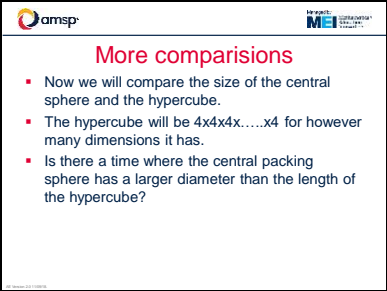
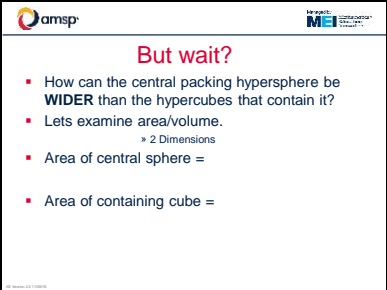
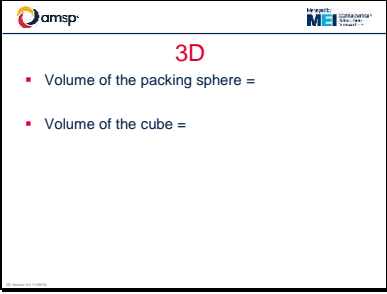
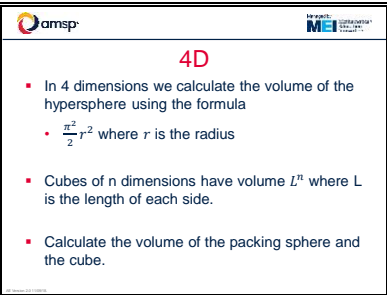


Slide 1	 <p>Advanced Mathematics Support Programme®</p>	
Slide 2	 <p>Spheres in higher dimensions</p>	
Slide 3	 <p>Packing spheres</p> <ul style="list-style-type: none"> Take a 4x4 square. Divide it into 4 2x2 squares. Put a circle in each square. What is the area of each circle? 	Keeping the radius at 1 simplifies the algebra that is performed later as an easier calculation than if we have a different sized square.
Slide 4	 <p>Left over space</p> <ul style="list-style-type: none"> Now draw a circle in the middle, that touches but does not cross the 4 original circles. What is the radius of that circle? What is the area of that circle? 	Students will need to use Pythagoras. Encourage them to draw diagrams.
Slide 5	 <p>Packing sphere</p> <p>What is the area of the blue circle?</p> 	$r = \sqrt{2} - 1. A = \pi(\sqrt{2} - 1)^2 \cong 0.539$

Slide 6	  <h3 style="text-align: center;">3 Dimensions</h3> <ul style="list-style-type: none"> Now make a 4x4x4 cube. Divide it in to 4 unit cubes Put a sphere in to each cube. What is the radius of each sphere? What is the volume of each sphere? 	See if students think the packing sphere will be bigger or smaller than the original spheres? They could calculate the ratio of the circle to the packing circle, and then see if they think the ratio of the sphere to packing sphere will be larger or smaller.																												
Slide 7	  <h3 style="text-align: center;">Left over space</h3> <ul style="list-style-type: none"> Now put a sphere in the central space, again the largest sphere possible that just touches each of the other spheres but does not cross. What is the radius of that sphere? <ul style="list-style-type: none"> Calculate the diagonal length of a cube. Use this to help you calculate the radius of the sphere. 																													
Slide 8	  <h3 style="text-align: center;">Going beyond 3D</h3> <ul style="list-style-type: none"> We can use Pythagoras to calculate the diagonal of a unit cube. Every time we add an extra dimension, we add an extra term to Pythagoras. Calculate the diagonals of the cubes going beyond 3D <table border="1" data-bbox="432 992 683 1120"> <thead> <tr> <th>Number of dimensions</th> <th>Length of diagonal</th> </tr> </thead> <tbody> <tr> <td>2</td> <td>$\sqrt{(1^2+1^2)} = \sqrt{2}$</td> </tr> <tr> <td>3</td> <td>$\sqrt{(1^2+1^2+1^2)} = \sqrt{3}$</td> </tr> <tr> <td>4</td> <td></td> </tr> <tr> <td>5</td> <td></td> </tr> <tr> <td>6</td> <td></td> </tr> <tr> <td>n</td> <td></td> </tr> </tbody> </table>	Number of dimensions	Length of diagonal	2	$\sqrt{(1^2+1^2)} = \sqrt{2}$	3	$\sqrt{(1^2+1^2+1^2)} = \sqrt{3}$	4		5		6		n		<table border="1" data-bbox="778 835 1353 1462"> <thead> <tr> <th>Number of dimensions</th> <th>Length of diagonal</th> </tr> </thead> <tbody> <tr> <td>2</td> <td>$\sqrt{(1^2 + 1^2)} = \sqrt{2}$</td> </tr> <tr> <td>3</td> <td>$\sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}$</td> </tr> <tr> <td>4</td> <td>$\sqrt{4} = 2$</td> </tr> <tr> <td>5</td> <td>$\sqrt{5}$</td> </tr> <tr> <td>6</td> <td>$\sqrt{6}$</td> </tr> <tr> <td>n</td> <td>\sqrt{n}</td> </tr> </tbody> </table>	Number of dimensions	Length of diagonal	2	$\sqrt{(1^2 + 1^2)} = \sqrt{2}$	3	$\sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}$	4	$\sqrt{4} = 2$	5	$\sqrt{5}$	6	$\sqrt{6}$	n	\sqrt{n}
Number of dimensions	Length of diagonal																													
2	$\sqrt{(1^2+1^2)} = \sqrt{2}$																													
3	$\sqrt{(1^2+1^2+1^2)} = \sqrt{3}$																													
4																														
5																														
6																														
n																														
Number of dimensions	Length of diagonal																													
2	$\sqrt{(1^2 + 1^2)} = \sqrt{2}$																													
3	$\sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}$																													
4	$\sqrt{4} = 2$																													
5	$\sqrt{5}$																													
6	$\sqrt{6}$																													
n	\sqrt{n}																													
Slide 9	  <h3 style="text-align: center;">Visualising higher dimensions</h3> <ul style="list-style-type: none"> Compare the sizes of the unit spheres with the packing sphere. What happens to the comparative sizes of the two spheres in the 4th dimension? What happens beyond 4D? 	4 th dimension – unit spheres have radius 1, packing sphere has radius 1. Hence beyond 4D the radius of the packing sphere is greater than the unit sphere containing it.																												

<p>Slide 10</p>		<p>Ensure students understand why this is a baffling concept. This means that the packing sphere that is being contained by the unit spheres that are within the cube is wider than the cube.</p> <p>You may want to reshew the diagram of the 3D cube with the packing sphere to emphasise the point.</p> <p>For diameter to be larger in length than the hypercube the radius needs to be bigger than 2.</p> <p>The radius equals 2 in the 9th dimension. So in 10D the diameter of the packing sphere is greater than the cube that contains it.</p>
<p>Slide 11</p>		<p>Central sphere = 0.539</p> <p>Cube = $4^2 = 16$</p> <p>The central sphere is significantly smaller than the containing cube.</p>
<p>Slide 12</p>		<p>Central sphere = 1.64</p> <p>Cube = 64</p>
<p>Slide 13</p>		<p>Sphere = 4.93</p> <p>Cube = 256</p> <p>Ask students what is happening to the comparative volume of the sphere and cubes as the dimensions increase.</p> <p>Ask them to compare this to their knowledge that the diameter of the sphere is becoming larger than the width of the cube.</p>

Slide 14

amsp

MEL

n Dimensions

- We're going to compare the volume of the sphere to the volume of the cube in more dimensions.
- Can you describe what
 - $\frac{V_{\text{sphere}}}{V_{\text{cube}}} > 1$,
 - $\frac{V_{\text{sphere}}}{V_{\text{cube}}} = 1$,
 - $\frac{V_{\text{sphere}}}{V_{\text{cube}}} < 1$
 would mean?

Slide 15

amsp

MEL

Comparing volumes

- We're going to compare the sum of all the volumes of the spheres, with the volume of the cube.

Slide 16

amsp

MEL

Making comparisons

Formula for volume of sphere	Dimension	Radius of packing sphere	Volume of packing sphere	Volume of Cube	$\frac{V_{\text{sphere}}}{V_{\text{cube}}}$
πr^2	2	$\sqrt{2} - 1$	0.539	$4^2 = 16$	0.0337
$\frac{4}{3}\pi r^3$	3	$\sqrt{3} - 1$	1.64	$4^3 = 64$	0.0257
$\frac{\pi^2}{2} r^4$	4	1			
$\frac{8\pi^2}{15} r^5$	5	$\sqrt{5} - 1$			
$\frac{\pi^3}{6} r^6$	6	$\sqrt{6} - 1$			

Slide 17

amsp

MEL

Answers on next slide







Slide 18

amsp

MEL

Making comparisons

Formula for volume of sphere	Dimension	Radius of packing sphere	Volume of packing sphere	Volume of Cube	$\frac{V_{\text{sphere}}}{V_{\text{cube}}}$
πr^2	2	$\sqrt{2} - 1$	0.539	$4^2 = 16$	0.0337
$\frac{4}{3}\pi r^3$	3	$\sqrt{3} - 1$	1.64	$4^3 = 64$	0.0257
$\frac{\pi^2}{2} r^4$	4	1	4.93	256	0.0193
$\frac{8\pi^2}{15} r^5$	5	$\sqrt{5} - 1$	15.2	1024	0.0148
$\frac{\pi^3}{6} r^6$	6	$\sqrt{6} - 1$	47.9	4096	0.0117

Slide 19	  <p style="text-align: center;">And...?</p> <ul style="list-style-type: none"> ▪ What is happening to the ratio of the volumes? ▪ What does this mean for the shape of hyperspheres and hypercubes? 	<p>The packing sphere is taking up a smaller and smaller part of the cube. If you look at the unit spheres, the same thing can be said (this is a good extension, students will have to think about how many packing spheres you fit in to each cube as you increase the dimensions (it doubles with each dimension)).</p>
Slide 20	  <p style="text-align: center;">Spiky spheres</p> <ul style="list-style-type: none"> ▪ As the volume of the packing sphere reduces as a proportion of the cube, and the cube takes up all the space it can, we can say that the cube is getting 'spiky'. ▪ This is how we can try to rationalise the sphere having a radius larger than the length of the containing box. ▪ However, the ratio starts to increase again, when the cube reaches the 1,206th dimension, the volume of the central packing sphere is greater than the volume of the cube! ▪ Hypergeometry is baffling! 	
Slide 21	  <p style="text-align: center;">Next</p> <ul style="list-style-type: none"> ▪ This activity was inspired by <ul style="list-style-type: none"> • This blog • This discussion and • This numberphile video 	