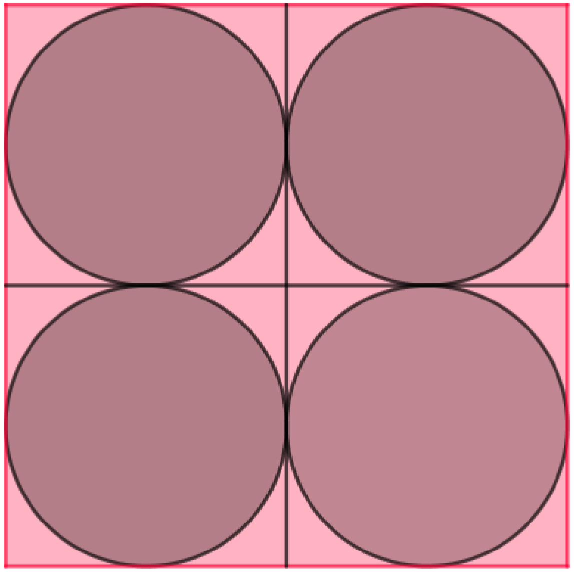
Advanced Mathematics Support Programme®

Spheres in higher dimensions

Packing spheres

- Take a 4x4 square.
- Divide it in to 4 2x2 squares.
- Put a circle in each square.
- What is the area of each circle?

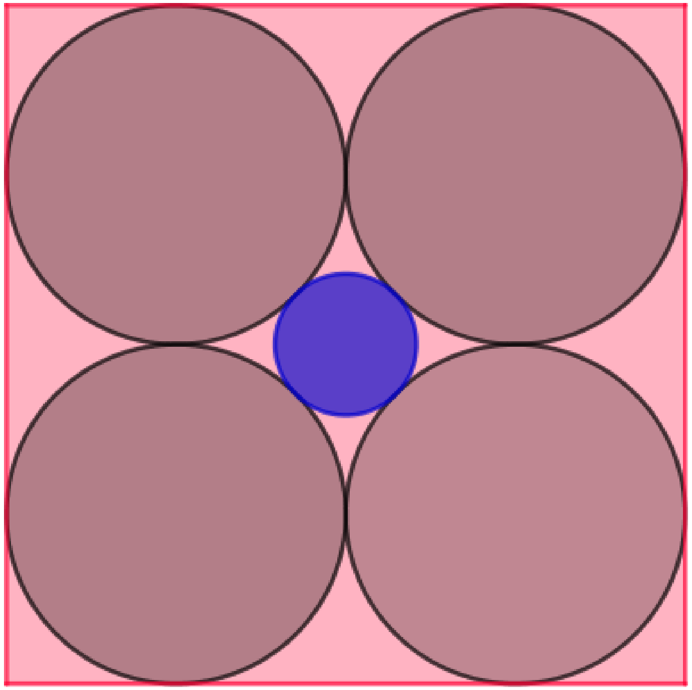


Left over space

- Now draw a circle in the middle, that touches but does not cross the 4 original circles.
- What is the radius of that circle?
- What is the area of that circle?

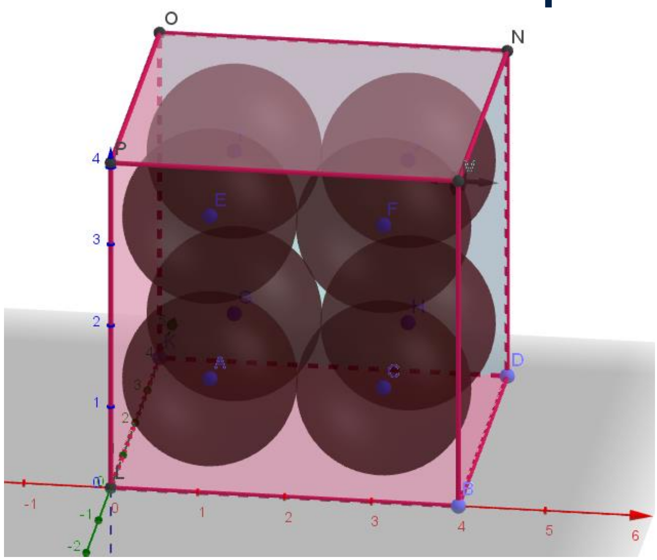
Packing sphere

What is the area of the blue circle?



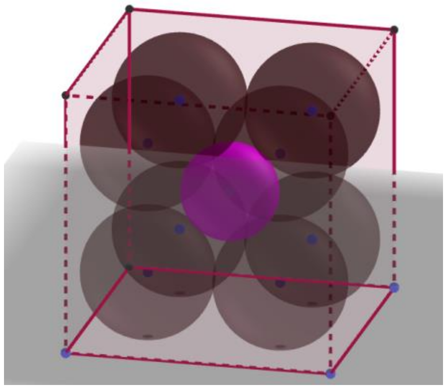
3 Dimensions

- Now make a 4x4x4 cube.
- Divide it in to 4 unit cubes
- Put a sphere in to each cube.
- What is the radius of each sphere?
- What is the volume of each sphere?



Left over space

- Now put a sphere in the central space, again the largest sphere possible that just touches each of the other spheres but does not cross.
- What is the radius of that sphere?
    - Calculate the diagonal length of a cube.
    - Use this to help you calculate the radius of the sphere.



Going beyond 3D

- We can use Pythagoras to calculate the diagonal of a unit cube.
- Every time we add an extra dimension, we add an extra term to Pythagoras.
- Calculate the diagonals of the cubes going beyond 3D

Number of dimensions	Length of diagonal
2	$\sqrt{(1^2+1^2)} = \sqrt{2}$
3	$\sqrt{(1^2+1^2+1^2)} = \sqrt{3}$
4	
5	
6	
n	

Visualising higher dimensions

- Compare the sizes of the unit spheres with the packing sphere.
- What happens to the comparative sizes of the two spheres in the 4th dimension?
- What happens beyond 4D?

More comparisions

- Now we will compare the size of the central sphere and the hypercube.
- The hypercube will be 4x4x4x…..x4 for however many dimensions it has.
- Is there a time where the central packing sphere has a larger diameter than the length of the hypercube?

But wait?

- How can the central packing hypersphere be **WIDER** than the hypercubes that contain it?
- Lets examine area/volume.

  » 2 Dimensions

- Area of central sphere =

- Area of surrounding cube =

3D

- Volume of the packing sphere =

- Volume of the cube =

4D

- In 4 dimensions we calculate the volume of the hypersphere using the formula
  - $\frac{\pi^2}{2}r^2$ where r is the radius

- Cubes of n dimensions have volume L^n where L is the length of each side.

- Calculate the volume of the packing sphere and the cube.

n Dimensions

- We're going to compare the volume of the sphere to the volume of the cube in more dimensions.
- Can you describe what
  - $\frac{V_{sphere}}{V_{cube}} > 1$,
  - $\frac{V_{sphere}}{V_{cube}} = 1$,
  - $\frac{V_{sphere}}{V_{cube}} < 1$

  would mean?

Comparing volumes

- We’re going to compare the sum of all the volumes of the spheres, with the volume of the cube.
- What patterns do you expect to happen?

Making comparisons

Formula for volume of sphere	Dimension	Radius of packing sphere	Volume of packing sphere	Volume of Cube	$\frac{V_{sphere}}{V_{cube}}$
πr^2	2	$\sqrt{2}-1$	0.539	$4^2 = 16$	0.0337
$\frac{4}{3}\pi r^3$	3	$\sqrt{3}-1$	1.64	$4^3 = 64$	0.0257
$\frac{\pi^2}{2}r^4$	4	1			
$\frac{8\pi^2}{15}r^5$	5	$\sqrt{5}-1$			
$\frac{\pi^3}{6}r^6$	6	$\sqrt{6}-1$			

Answers on next slide

Making comparisons

Formula for volume of sphere	Dimension	Radius of packing sphere	Volume of packing sphere	Volume of Cube	$\frac{V_{sphere}}{V_{cube}}$
πr^2	2	$\sqrt{2}-1$	0.539	$4^2 = 16$	0.0337
$\frac{4}{3}\pi r^3$	3	$\sqrt{3}-1$	1.64	$4^3 = 64$	0.0257
$\frac{\pi^2}{2}r^4$	4	1	4.93	256	0.0193
$\frac{8\pi^2}{15}r^5$	5	$\sqrt{5}-1$	15.2	1024	0.0148
$\frac{\pi^3}{6}r^6$	6	$\sqrt{6}-1$	47.9	4096	0.0117

And…?

- What is happening to the ratio of the volumes?
- What does this mean for the shape of hyperspheres and hypercubes?

Spiky spheres

- As the volume of the packing sphere reduces as a proportion of the cube, and the cube takes up all the space it can, we can say that the cube is getting ‘spiky’.
- This is how we can try to rationalise the sphere having a radius larger than the length of the containing box.
- However, the ratio starts to increase again, when the cube reaches the 1,206th dimension, the volume of the central packing sphere is greater than the volume of the cube!
- Hypergeometry is baffling!

Next

- This activity was inspired by
  - This blog
  - This discussion and
  - This numberphile video