Advanced Mathematics
Support Programme®

About the AMSP

- A government-funded initiative, managed by MEI, providing national support for teachers and students in all state-funded schools and colleges in England.
- It aims to increase participation in AS/A level Mathematics and Further Mathematics, and Core Maths, and improve the teaching of these qualifications.
- Additional support is given to those in priority areas to boost social mobility so that, whatever their gender, background or location, students can choose their best maths pathway post-16, and have access to high quality maths teaching.

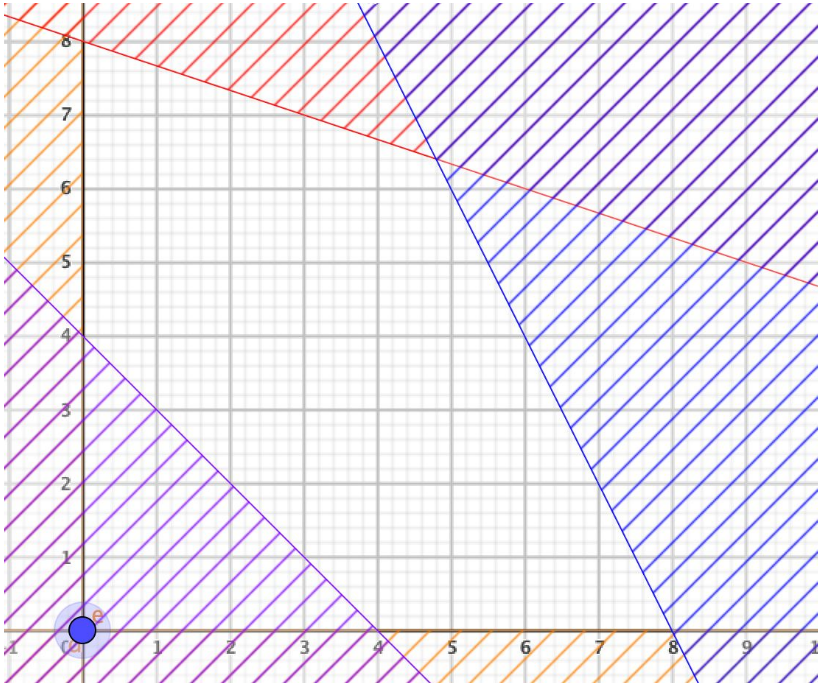
MEI holds the NCETM CPD Standard

The CPD Standard supports maths teachers to access information about the wide range of CPD provision on offer and to be assured of its appropriateness and quality.

ncetm.org.uk/cpdstandard

Continuing Professional Development Standard

National Centre for Excellence in the Teaching of Mathematics



2 Stage Simplex

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To think about:

In the example on the left, the origin isn't on the boundary of the feasible region.

Starting at the origin, how do we get to the boundary so that we can apply the simplex algorithm?

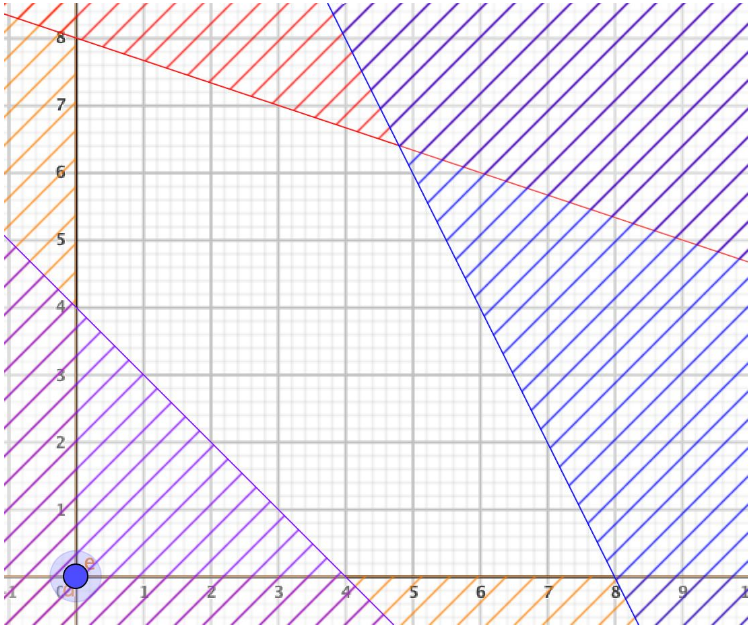
Discrete/Decision Mathematics Topics	AQA	Edexcel	MEI	OCR A
Formulating constrained problems into Linear programs	**AS**	**AS D1**	**MwA**	**AS**
Graphical solution using an objective function	**AS**	**AS D1**	**MwA**	**AS**
Integer solution		**AS D1**	**MwA**	A Level
Slack variables	A Level	A Level D1	**MwA**	A Level
Simplex Method	A Level	A Level D1	**MwA**	A Level
Interpretation of Simplex	A Level	A Level D1	**MwA**	A Level
Big M method		A Level D1		
Integer programming, branch-and-bound method				A Level
Post-optimal analysis			**MwA**	A Level
Formulate a range of network problems as LPs			**MwA**	
Use of software and interpretation of output			**MwA**	

In this session:

- ≥ constraints
- Artificial variables
- Surplus variables
- Two stage Simplex
- Playtime!
- Technology

Two stage simplex

- The simplex algorithm relies on (0,0) being a feasible solution.
  - This isn't possible if there are ≥ constraints (we maintain the trivial constraints eg $y \geq 0$)



Slack and Surplus

- For ≤ constraints we add slack variables (how much is '*missing*')
- For ≥ constraints we subtract ***surplus*** variables (how much '*extra*' we have).
- As with other constraints these will be positive.

Example

Maximise $P = x + 0.8y$

Subject to

$x + y \le 1000$, $2x + y \le 1500$, $3x + 2y \le 2400$

$x + y \ge 800$, $x, y \ge 0$

$P - x - 0.8y = 0$

$x + y + s_1 = 1000$

$2x + y + s_2 = 1500$

$3x + 2y + s_3 = 2400$

$x + y - s_4 = 800$

$x, y, s_1, s_2, s_3, s_4 \ge 0$

Example

Maximise $P = x + 0.8y$

Subject to

$x + y \le 1000$, $2x + y \le 1500$, $3x + 2y \le 2400$

$x + y \ge 800$, $x, y \ge 0$

$P - x - 0.8y = 0$

$x + y + s_1 = 1000$

$2x + y + s_2 = 1500$

$3x + 2y + s_3 = 2400$

$x + y - s_4 = 800$

$x, y, s_1, s_2, s_3, s_4 \ge 0$

s_4 is a surplus variable

Example

Maximise P= x+0.8y

Subject to

$x + y \le 1000, \quad 2x + y \le 1500, 3x+2y \le 2400$

$x + y \ge 800, \quad x, y \ge 0$

$P - x - 0.8y = 0$

$x + y + s_1 = 1000$

$2x + y + s_2 = 1500$

$3x + 2y + s_3 = 2400$

$x + y - s_4 = 800$

$x, y, s_1, s_2, s_3, s_4 \ge 0$

Example

Maximise P= $x+0.8y$

Subject to

$x + y \le 1000$, $2x + y \le 1500$, $3x+2y \le 2400$

$x + y \ge 800$, $x, y \ge 0$

$P - x - 0.8y = 0$

$x + y + s_1 = 1000$

$2x + y + s_2 = 1500$

$3x + 2y + s_3 = 2400$

$x + y - s_4 + a_1 = 800$

$x, y, s_1, s_2, s_3, s_4, a_1 \ge 0$

Add artificial variables with each surplus variable

$x + y - s_4 + a_1 = 800$

Example

Maximise $P = x + 0.8y$

Subject to

$x + y \le 1000$, $2x + y \le 1500$, $3x + 2y \le 2400$

$x + y \ge 800$, $x, y \ge 0$

$P - x - 0.8y = 0$

$x + y + s_1 = 1000$ $= 1000$

$2x + y + s_2 = 1500$ $= 1500$

$3x + 2y +$ …

$x + y - s_4 + a_1 = 800$ s_4 $a_1 = 800$

$x, y, s_1, s_2, s_3, s_4, a_1 \ge 0$

Two stage simplex

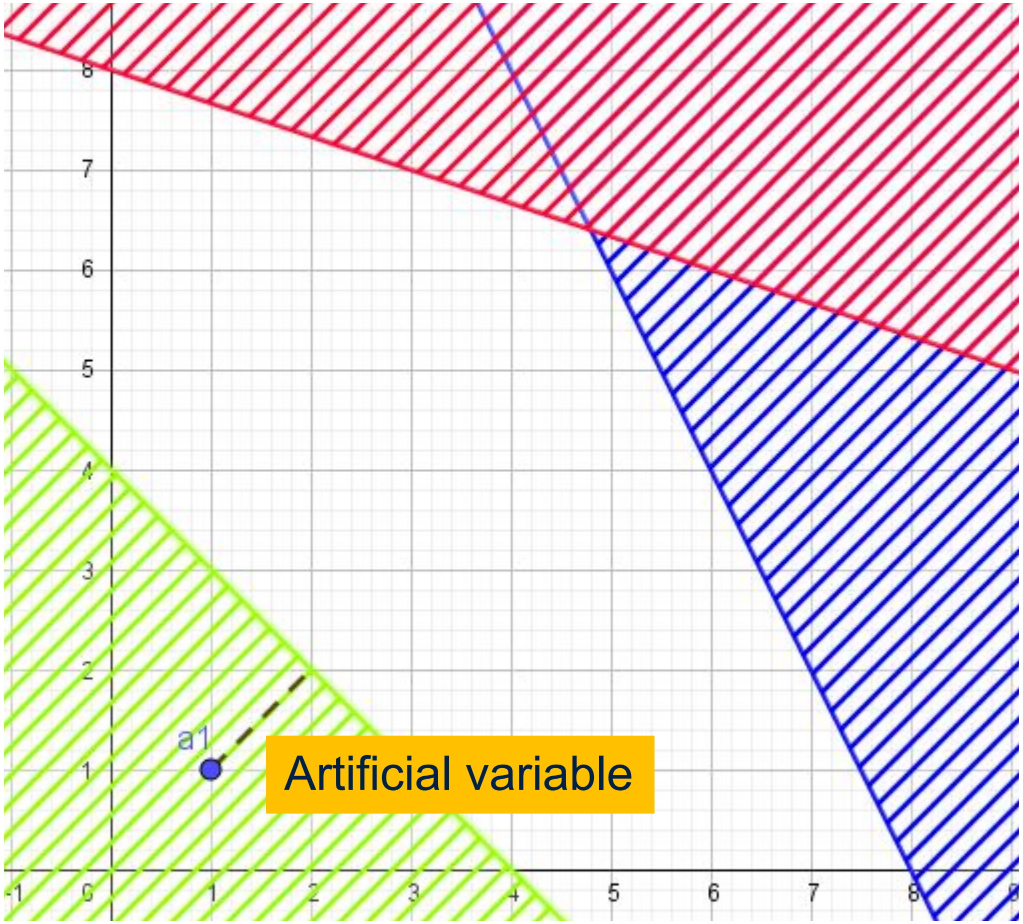
- The simplex algorithm relies on (0,0) being a feasible solution.
- If there are ≥ constraints, add artificial variables and subtract surplus variables
- Two stage simplex has a second objective that is equal to the sum of the artificial variables
- The first stage makes the second objective zero, the second stage is as normal

Greater than or equal to constraints

- Simplex always starts at the origin.
- If there are ≥ constraints, add artificial variables introduce a new objective function

  $\boldsymbol{A = a_1 + a_2 + \ldots}$

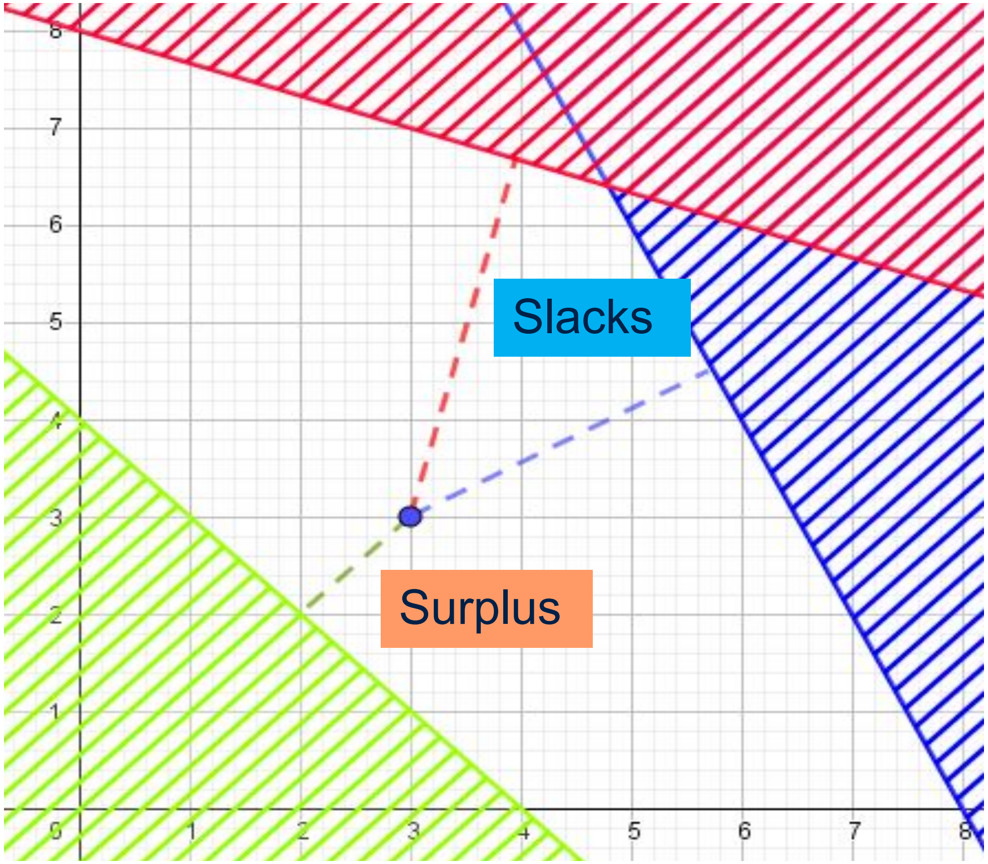
  which you must minimise, since when $A = 0$ you are in the feasible region



Greater than or equal to constraints

Once in the feasible region:

- **Surplus variables** give a measure of the perpendicular distance from each ≥ inequality
- **Slack variables** give a measure of the perpendicular distance from each ≤ inequality.
- Complete the simplex in the normal way.



Two Stage Simplex

Maximise P= x + 0.8y

Subject to

$x + y \leq 1000$, $2x + y \leq 1500$, $3x + 2y \leq 2400$

$x + y \geq 800$

$P - x - 0.8y = 0$

$x + y + s_1 = 1000$

$2x + y + s_2 = 1500$

$3x + 2y + s_3 = 2400$

$x + y - s_4 + a_1 = 800$

Subtract surplus variables and add artificial variables

Two Stage Simplex

Maximise $P - x - 0.8y = 0$,

Subject to $x + y + s_1 = 1000$

$2x + y + s_2 = 1500$, $3x + 2y + s_3 = 2400$

$x + y - s_4 + a_1 = 800$

Sometimes this is '*I*' *Check with your board*

Minimise $A = a_1 + a_2 + a_3 + \ldots$

If $A = 0$ we get back to the boundary of the feasible region.

but we write this in terms of the other variables:

$$A = -x - y + s_4 + 800$$

$$A + x + y - s_4 = 800$$

Difference in specs

- Some specs wish you to:
  - Minimise $A = a_1 + a_2 + a_{3\,+\,\ldots}$ (eg MEI)
  - Maximise $I = -(a_1 + a_2 + a_{3\,+\,\ldots})$ (eg Edexcel)
- This is effectively the same, but you should know which you board expects
- In the tableaux it will adjust how you select the pivot column:
  - A positive number if minimising (eg MEI)
  - A negative number if maximising (eg Edexcel)

Two Stage Simplex

Maximise $P - x - 0.8y = 0$,

Subject to $x + y + s_1 = 1000$

$2x + y + s_2 = 1500$, $3x + 2y + s_3 = 2400$

$x + y - s_4 + a_1 = 800$

Simplex spreadsheet

Minimise $A = a_1 + a_2 + a_{3} + \ldots$

but we write this in terms of the other variables:

$$A = -x - y + s_4 + 800$$

$$A + x + y - s_4 = 800$$

SIMPLEX 2 Maximising subject to ≤ and/or ≥ constraints

Setup | Reset | Auto | 1

No. variables: 2 | No. ≤ constraints: 3 | No. ≥ constraints: 1

Stage 1: Pivot 1 | Pivot 2 | Pivot 3 | Pivot 4

Stage 2: Pivot 1 | Pivot 2 | Pivot 3 | Pivot 4

Initial Tableau

A	P	x	y	s1	s2	s3	s4	a1	RHS	Min ratio
1	0	1	1	0	0	0	-1	0	800	
0	1	-1	-0.8	0	0	0	0	0	0	
0	0	1	1	1	0	0	0	0	1000	
0	0	2	1	0	1	0	0	0	1500	750
0	0	3	2	0	0	1	0	0	2400	
0	0	1	1	0	0	0	-1	1	800	

Stage 1 Pivot 1

A	P	x	y	s1	s2	s3	s4	a1	RHS	Min ratio
1	0	0	0.5	0	-0.5	0	-1	0	50	
0	1	0	-0.3	0	0.5	0	0	0	750	
0	0	0	0.5	1	-0.5	0	0	0	250	
0	0	1	0.5	0	0.5	0	0	0	750	
0	0	0	0.5	0	-1.5	1	0	0	150	
0	0	0	0.5	0	-0.5	0	-1	1	50	100

Stage 1 Pivot 2

A	P	x	y	s1	s2	s3	s4	a1	RHS	Min ratio
1	0	0	0	0	0	0	0	-1	0	
0	1	0	0	0	0.2	0	-0.6	0.6	780	
0	0	0	0	1	0	0	1	-1	200	
0	0	1	0	0	1	0	1	-1	700	
0	0	0	0	0	-1	1	1	-1	100	100
0	0	0	1	0	-1	0	-2	2	100	

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SIMPLEX 2 Maximising subject to ≤ and/or ≥ constraints

Setup | Reset | Auto

No. variables: 2

No. ≤ constraints: 3

No. ≥ constraints: 1

Stage 1: Pivot 1 | Pivot 2 | Pivot 3 | Pivot 4

Stage 2: Pivot 1 | Pivot 2 | Pivot 3 | Pivot 4

Stage 1 Pivot 2

A	P	x	y	s1	s2	s3	s4	a1	RHS	Min ratio
1	0	0	0	0	0	0	0	-1	0	
0	1	0	0	0	0.2	0	-0.6	0.6	780	
0	0	0	0	1	0	0	1	-1	200	
0	0	1	0	0	1	0	1	-1	700	
0	0	0	0	0	-1	1	1	-1	100	100
0	0	0	1	0	-1	0	-2	2	100	

Stage 2 Pivot 1

A	P	x	y	s1	s2	s3	s4	a1	RHS	Min ratio
0	1	0	0	0	-0.4	0.6	0	0	840	
0	0	0	0	1	1	-1	0	0	100	100
0	0	1	0	0	2	-1	0	0	600	
0	0	0	0	0	-1	1	1	-1	100	
0	0	0	1	0	-3	2	0	0	300	

Stage 2 Pivot 2

A	P	x	y	s1	s2	s3	s4	a1	RHS
0	1	0	0	0.4	0	0.2	0	0	880
0	0	0	0	1	1	-1	0	0	100
0	0	1	0	-2	0	1	0	0	400
0	0	0	0	1	0	0	1	-1	200
0	0	0	1	3	0	-1	0	0	600

Solution

40									
41	Stage 2		Pivot	2					
42	A	P	x	y	s1	s2	s3	s4	RHS
43									
44	0	1	0	0	0.4	0	0.2	0	880
45	0	0	0	0	1	1	-1	0	100
46	0	0	1	0	-2	0	1	0	400
47	0	0	0	0	1	0	0	1	200
48	0	0	0	1	3	0	-1	0	600

So the solution is

x = 400

y = 600

P = 880

s_2 = 100

Note that s_4 is also basic meaning that the solution does not lie on an intersection with the ≥ constraint

Edexcel Tableaux

Row Ops	B.V.	x	y	S_1	S_2	S_3	S_4	a_1	RHS	θ
$R_1 = R_1$	S_4	0	0	1	0	0	1		200	-
$R_2 = R_2 - R_1$	S_2	1	0	-1	1	0	0		500	500
$R_3 = R_3 - 2R_1$	S_3	1	0	-2	0	1	0		400	400
$R_4 = R_4 + R_1$	y	1	1	1	0	0	0		1000	1000
$R_5 = R_5 + 0.8R_1$	P	- 1/5	0	4/5	0	0	0		800	

Minimising Problems

Minimise $C = -4x + y$

Subject to:

$$-3x + 2y \leq 6$$
$$x \leq 3$$
$$3x + y \geq 6$$
$$x, y \geq 0$$

Minimise $C = -4x + y$

$\Longrightarrow$ Maximise $P = -C = 4x - y$

	BV	x	y	z	s_1	s_2	s_3	a_1	RHS	θ.
$R_1 = R_1 + R_2$	s_1	0	$8/3$		1	3	0	0	15	
$R_2 = 3R_2$	s_3	0	$-4/3$		0	3	1	-1	3	
$R_3 = R_3 + 1/3 R_2$	x	1	0		0	1	0	0	3	
$R_4 = R_4 + 4/3 R_2$	P	0	$13/9$		0	30	0	0	12	

Max $P = 12$

$\therefore$ min $C = -12$, $x = 3$, $s_1 = 15$, $s_3 = 3$.

- Maximise $P = -C$
- $\therefore C = -P = 12$

SAMs

Three liquid medicines, X, Y and Z, are to be manufactured. All the medicines require ingredients A, B, C and D which are in limited supply. The table below shows how many grams of each ingredient are required for one litre of each medicine. It also shows how much of each ingredient is available.

	A	B	C	D
Each litre of X requires	2	0	2	4
Each litre of Y requires	5	2	4	3
Each litre of Z requires	3	1	2	2
Amount, in grams, of each ingredient available	20	10	70	30

When the medicines are sold, the profits are £5 per litre of X manufactured, £2 per litre of Y and £3 per litre of Z.

(i) Formulate an LP to maximise the total profit subject to the constraints imposed by the availability of the ingredients. Use x as the number of litres of X, y as the number of litres of Y and z as the number of litres of Z. **[3]**

When the medicines are sold, the profits are £5 per litre of X manufactured, £2 per litre of Y and £3 per litre of Z.

(i) Formulate an LP to maximise the total profit subject to the constraints imposed by the availability of the ingredients. Use x as the number of litres of X, y as the number of litres of Y and z as the number of litres of Z. **[3]**

The simplex algorithm is used to solve this LP. After the first iteration the tableau below is produced.

P	x	y	z	s_1	s_2	s_3	s_4	RHS
1	0	1.75	–0.5	0	0	0	1.25	37.5
0	0	3.5	2	1	0	0	–0.5	5
0	0	2	1	0	1	0	0	10
0	0	2.5	1	0	0	1	–0.5	55
0	1	0.75	0.5	0	0	0	0.25	7.5

(ii) (*A*) Perform a second iteration. **[2]**

(*B*) Give the maximum profit, and the number of litres of X, Y and Z which should be manufactured to achieve this profit. **[1]**

(iii) An extra constraint is imposed by a contract to supply at least 5 litres of Y. Produce an initial tableau which could be used to solve this new problem by using the two-stage simplex method. **[3]**

Question			Answer	Marks	AO	Guidance
6	(i)		Maximise $5x+2y+3z$ subject to $2x+5y+3z \le 20$ $2y+z \le 10$ $2x+4y+2z \le 70$ $4x+3y+2z \le 30$ $x, y, z \ge 0$	M1 A2 [3]	3.3 3.3 1.1	objective constraints -1 each error
6	(ii)	(*A*)	See tableau below	M1 A1 [2]	3.4 1.1	Pivot all correct
6	(ii)	(*B*)	Maximum profit of £38.75 manufacturing 6.25 litres of X, none of Y and 2.5 litres of Z.	B1 [1]	3.2a	
6	(iii)		See tableau below	B1 B1 B1 [3]	 3.3 3.5c 1.1	 Surplus Additional variable New objective

6 (ii) (*A*):

P	x	y	z	s_1	s_2	s_3	s_4	RHS
1	0	2.625	0	0.25	0	0	1.125	38.75
0	0	1.75	1	0.5	0	0	–0.25	2.5
0	0	0.25	0	–0.5	1	0	0.25	7.5
0	0	0.75	0	–0.5	0	1	–0.25	52.5
0	1	– 0.125	0	–0.25	0	0	0.375	6.25

6 (iii):

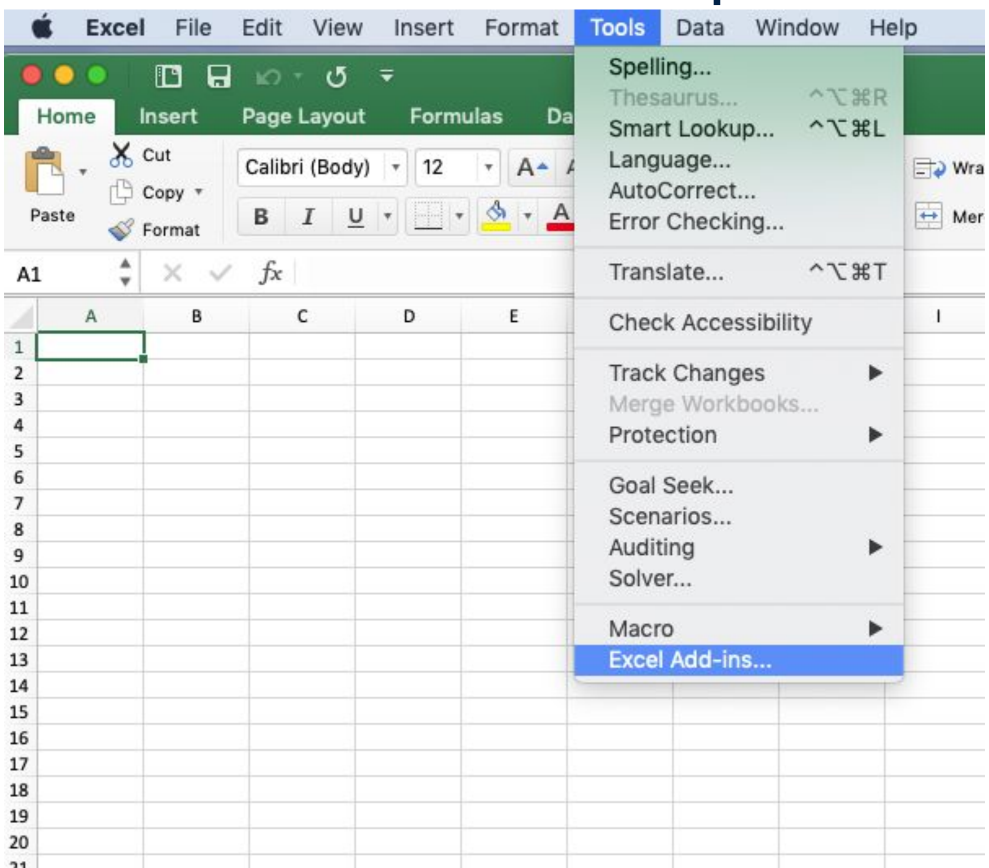
Q	P	x	y	z	s_1	s_2	s_3	s_4	s_5	a_5	RHS
1	0	0	1	0	0	0	0	0	–1	0	5
0	1	–5	–2	–3	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	–1	1	5
0	0	2	5	3	1	0	0	0	0	0	20
0	0	0	2	1	0	1	0	0	0	0	10
0	0	2	4	2	0	0	1	0	0	0	70
0	0	4	3	2	0	0	0	1	0	0	30

FM Videos

- Includes minimising problems
- 2 Stage
- Big M (Edexcel)
- Use of technology (MEI)

Using Excel ☺

- Mac: Tools > excel add ins > solver add in
- Windows: File > options > solver add in



Max $P = 4x - 5y + z$

s.t.

$3x + 4z \leq 24$

$x + y \leq 7$

$x \leq 6$

Initial values (the origin)

	A	B	C	D	E
1	x	y	z	P	
2	0	0	0		
3	4	-5	1	0	
4	3	0	4	0	24
5	1	1	0	0	7
6	1	0	0	0	6

Objective values are not taken to the LHS

This calculates the profit at each vertex being considered

Ie $P = 4x - y + z$.

TRANSPOS =SUMPRODUCT(A$2:C$2,A3:C3)

	A	B	C	D	E
1	x	y	z	P	
2	0	0	0		
3	4	-5	1	C$2,A3:C3)	
4	3	0	4	0	24
5	1	1	0	0	7
6	1	0	0	0	6

Now consider the LHS of the constraints.

Cells D4, D5, D6 calculate the value of the constraints at the point being considered.

TRANSPOS =SUMPRODUCT(A$2:C$2,A4:C4)

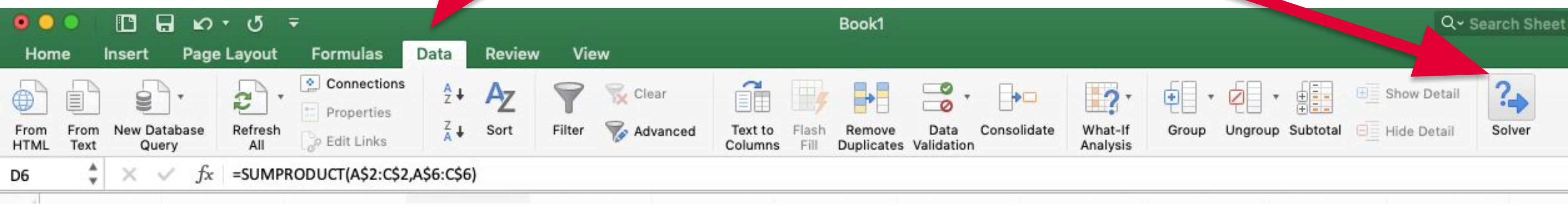
	A	B	C	D	E
1	x	y	z	P	
2	0	0	0		
3	4	-5	1	0	
4	3	0	4	C$2,A4:C4)	24
5	1	1	0	0	7
6	1	0	0	0	6

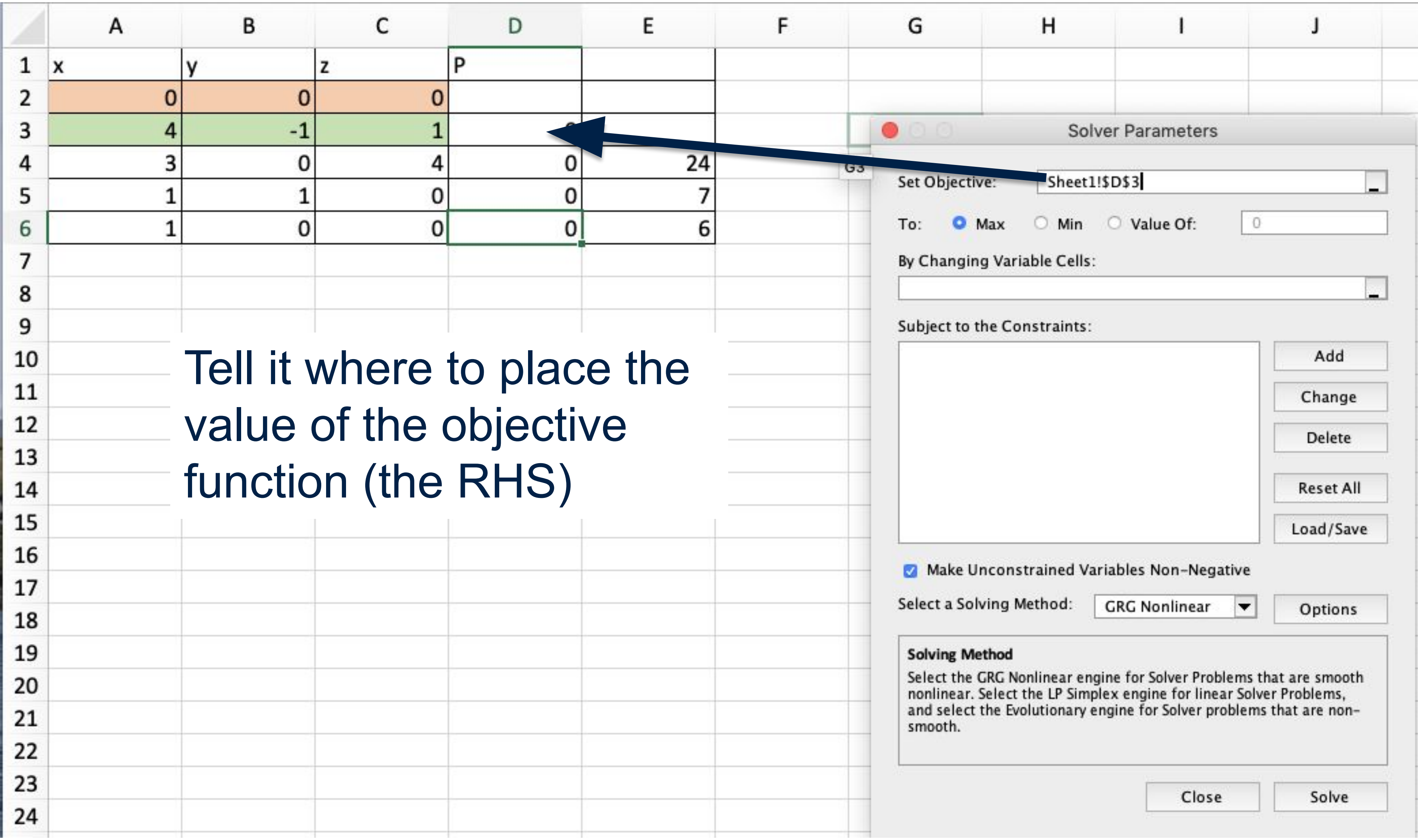
D5 =SUMPRODUCT(A$2:C$2,A5:C5)

D6 =SUMPRODUCT(A$2:C$2,A6:C6)

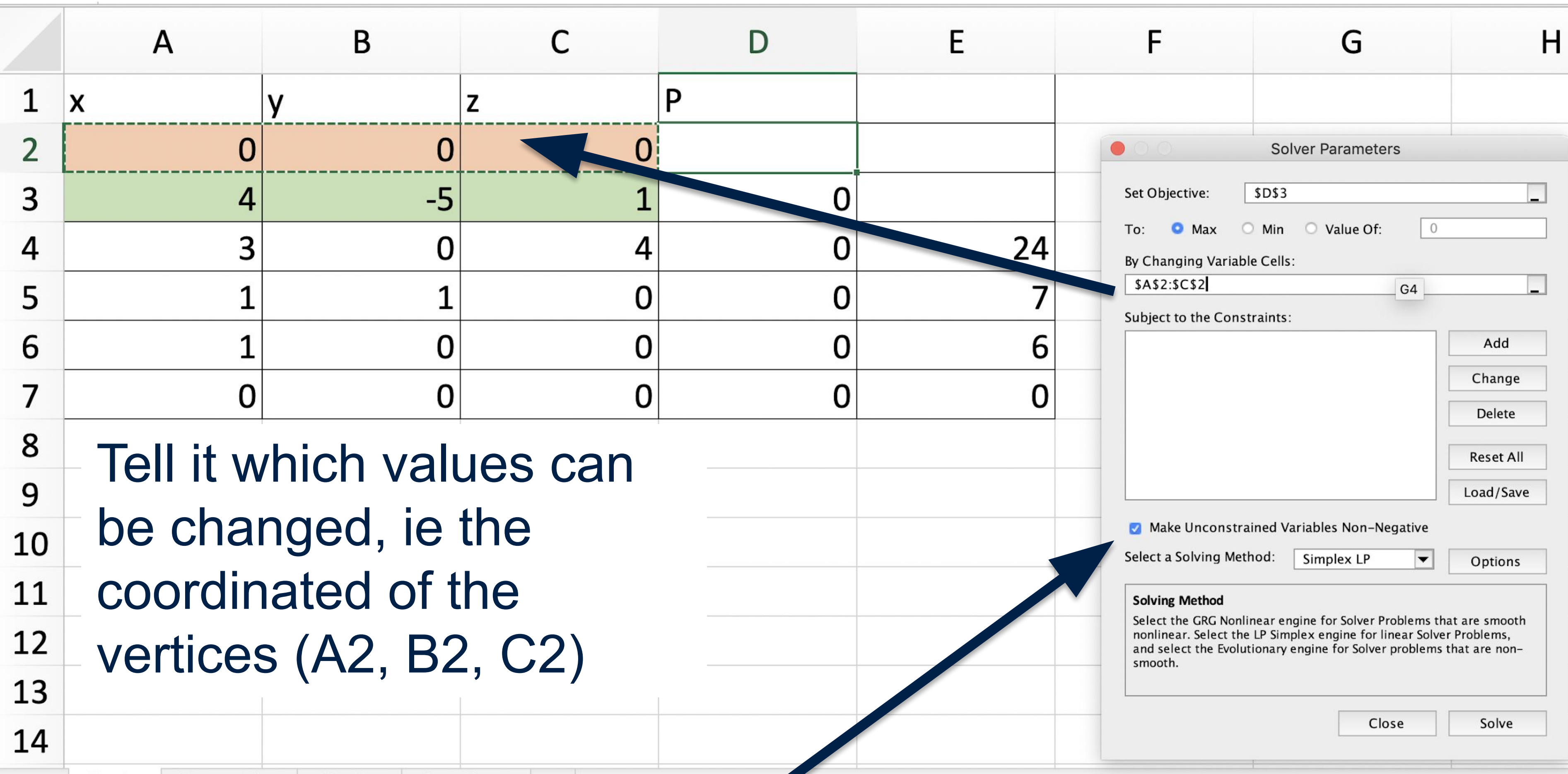
Now we can use the solver!

- Go to the data tab > solver



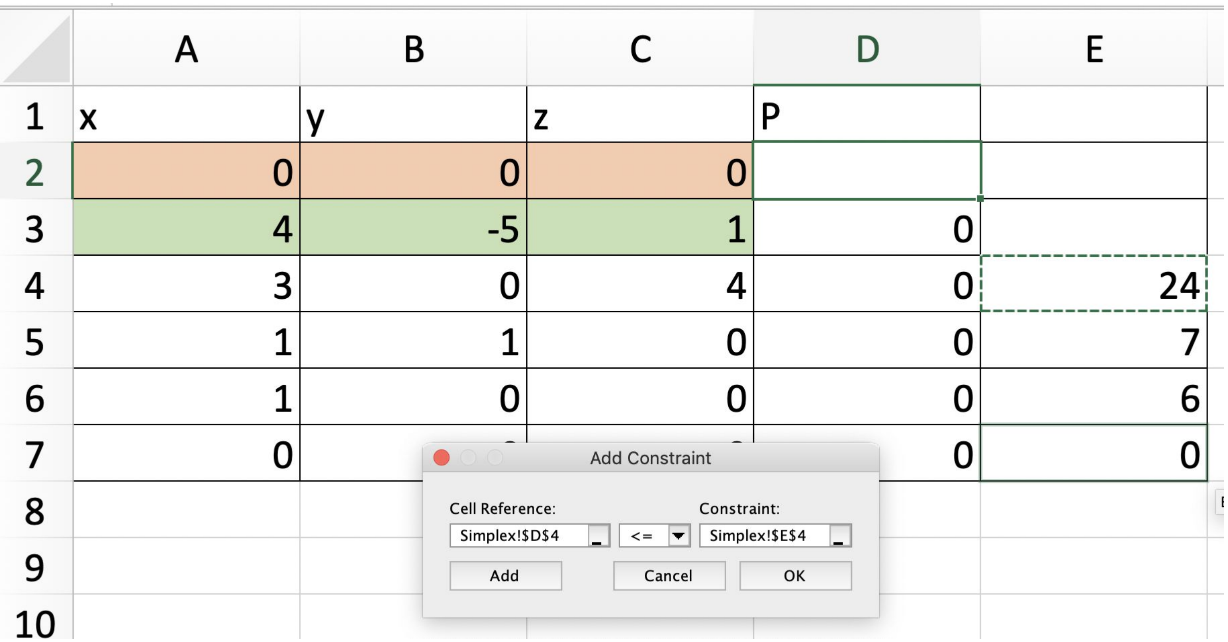


Tell it where to place the value of the objective function (the RHS)



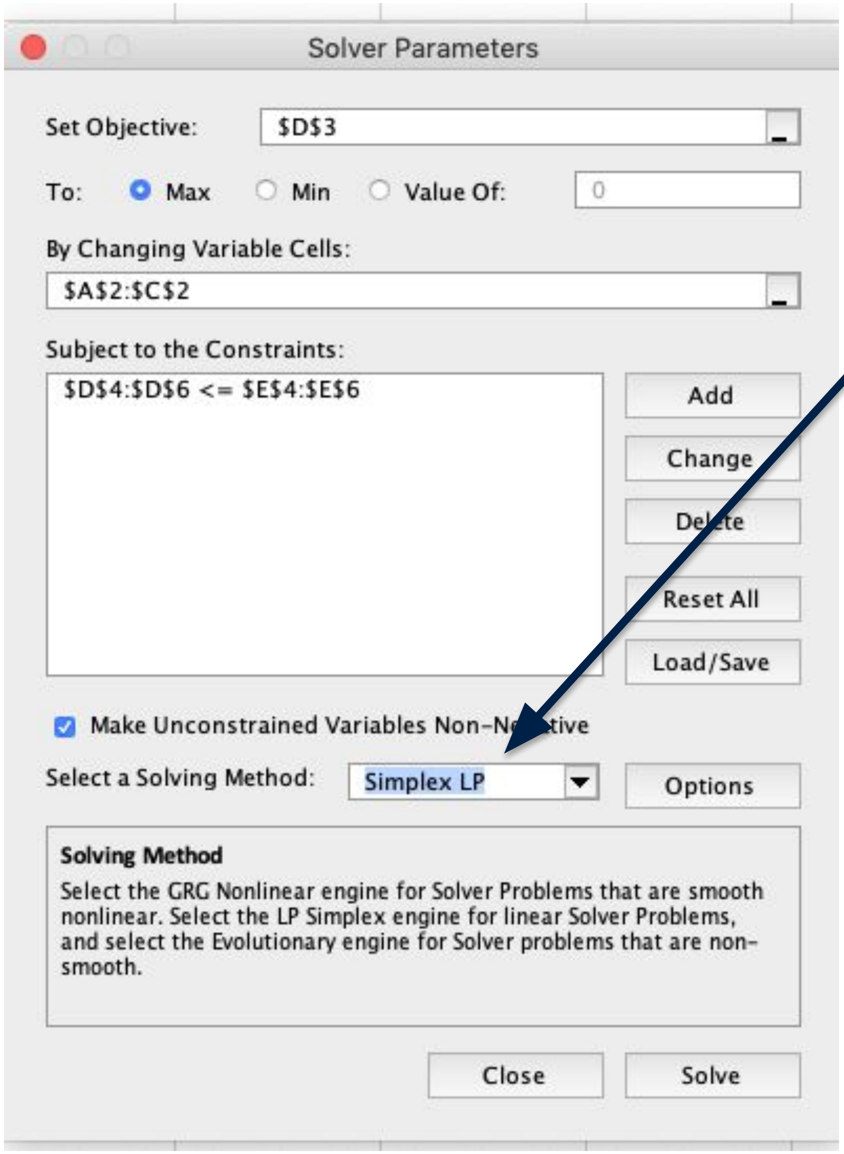
Tell it which values can be changed, ie the coordinated of the vertices (A2, B2, C2)

Non negativity constraints

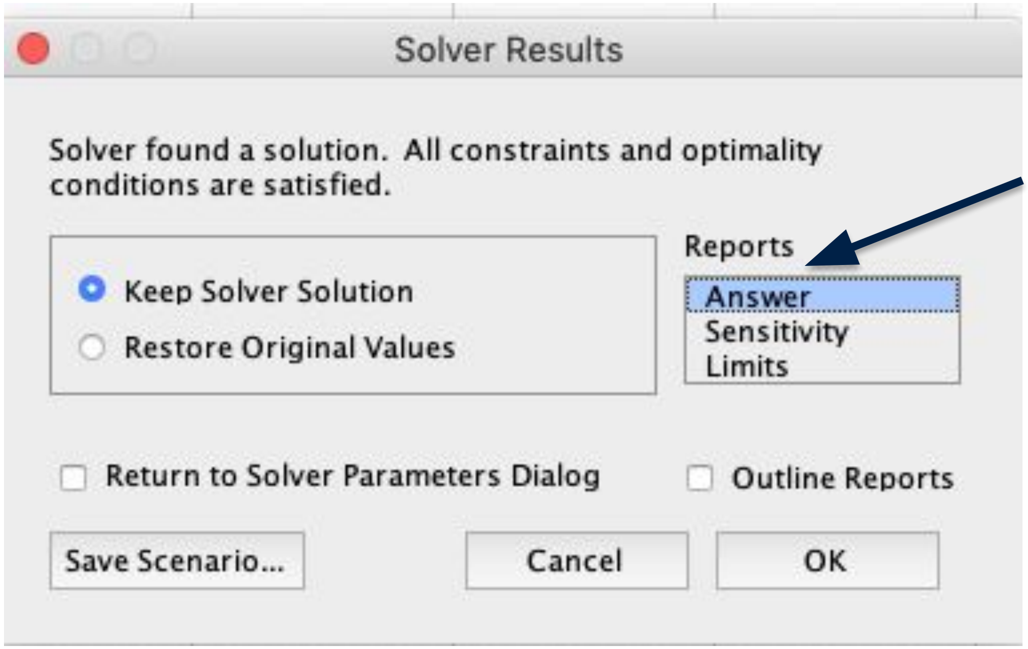


For the constraints click ‘add’ to get this screen.

Cell reference refers to cells D4 → D6, constraints to cells E4 → E6.



Don’t forget to tell it to use the simplex algorithm!



Click answer for the details (final profit, values (x, y, z, slack etc) to be stored in a new sheet).

Then click ok ☺

Microsoft Excel 16.14 Answer Report

Worksheet: [Book1]Sheet1

Report Created: 30/05/2019 12:39:45

Result: Solver found a solution. All constraints and optimality conditions are satisfied.

Solver Engine

Engine: Simplex LP

Solution Time: 369367783.863 Seconds.

Iterations: 1 Subproblems: 2

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling

Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$3	P	0	25

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$A$2	x	0	6	Integer
$B$2	y	0	0	Integer
$C$2	z	0	1	Integer

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$4	P	22	$D$4<=$E$4	Not Binding	2
$D$5	P	6	$D$5<=$E$5	Not Binding	1
$D$6	P	6	$D$6<=$E$6	Binding	0
$A$2:$C$2=Integer					

$P = 25$

$x = 6$

$y = 0$

$z = 0$

Slack

It’ll also work for two Stage

Max: $P - x - 0.8y = 0$,

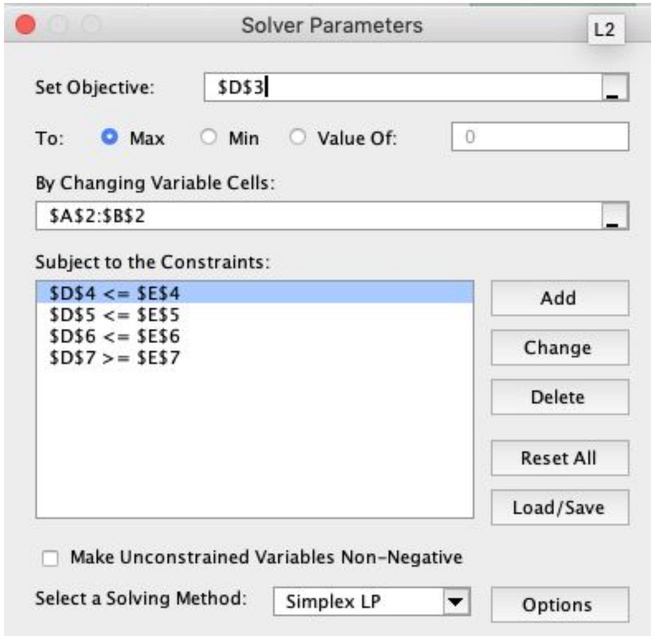
s.t. $x + y + s_1 = 1000$

$2x + y + s_2 = 1500$,

$3x + 2y + s_3 = 2400$

$x + y - s_4 + a_1 = 800$

	A	B	D	E
1	x	y	P	
2	0	0		
3	1	0.8	0	
4	1	1	0	1000
5	2	1	0	1500
6	3	2	0	2400
7	1	1	0	800



	A	B	D	E
1	x	y	P	
2	400	600		
3	1	0.8	880	
4	1	1	1000	1000
5	2	1	1400	1500
6	3	2	2400	2400
7	1	1	1000	800

ive Cell (Max)

l	Name	Original Value	Final Value
3	P	0	880

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$A$2	x	0	400	Contin
$B$2	y	0	600	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$4	P	1000	$D$4<=$E$4	Binding	0
$D$5	P	1400	$D$5<=$E$5	Not Binding	100
$D$6	P	2400	$D$6<=$E$6	Binding	0
$D$7	P	1000	$D$7>=$E$7	Not Binding	200

Slack 2 (s_2) → 100

Surplus (s_4) → 200

http://www.zweigmedia.com/RealWorld/simplex.html

- Will do maximise and minimise problems
- Can't do integer programming
- Scrollable box
- Easy to input data

Type your linear programming problem below. (Press "Example" to see how to set it up.)

```
maximise P = x + 0.8y
subject to
x + y <= 1000
2x + y <= 1500
3x + 2y <= 2400
x + y >= 800
```

Solution:

Optimal Solution: p = 880; x = 400, y = 600

Solve | Example | Erase Everything | Rounding: 6 significant digits

Mode: Decimal / Fraction / Integer

The tableaus will appear here.

Tableau #1

x	y	s1	s2	s3	s4	p	
1	1	1	0	0	0	0	1000
2	1	0	1	0	0	0	1500
3	2	0	0	1	0	0	2400
1	1	0	0	0	-1	0	800
-1	-0.8	0	0	0	0	1	0

Tableau #2

x	y	s1	s2	s3	s4	p	
0	0.5	1	-0.5	0	0	0	250
1	0.5	0	0.5	0	0	0	750
0	0.5	0	-1.5	1	0	0	150
0	0.5	0	-0.5	0	-1	0	50
0	-0.3	0	0.5	0	0	1	750

Tableau #3

x	y	s1	s2	s3	s4	p	
0	0	1	0	0	1	0	200
1	0	0	1	0	1	0	700
0	0	0	-1	1	1	0	100
0	1	0	-1	0	-2	0	100
0	0	0	0.2	0	-0.6	1	780

Tableau #4

x	y	s1	s2	s3	s4	p	
0	0	1	1	-1	0	0	100
1	0	0	2	-1	0	0	600
0	0	0	-1	1	1	0	100
0	1	0	-3	2	0	0	300
0	0	0	-0.4	0.6	0	1	840

Tableau #5

x	y	s1	s2	s3	s4	p	
0	0	1	1	-1	0	0	100
1	0	-2	0	1	0	0	400
0	0	1	0	0	1	0	200
0	1	3	0	-1	0	0	600
0	0	0.4	0	0.2	0	1	880

Modelling

- Consider:
  - Old spec Edexcel D2 – transportation, allocation and game theory as LPP
  - Old spc MEI Discrete Computing – examples of outputs

Exam advice: simplex

- Many students can apply simplex accurately
- Some candidates fail to identify variables; there are still a number of candidates who define variables as 'a is crop A' etc,
- Most students can explain given inequalities but have more problems if asked to work them out from information given.
- Interpretation of solution is usually less good. Listing the values taken by the variables does not constitute interpretation, examiners needed to know what was to be made at what profit, and what would be left over.

Exam tips

- The most common mistakes are arithmetic.
- Number the rows and write the row operations being used at the side (e.g. R4-3R2). This helps both student and the examiner to keep track of what is happening.
- Know the conditions for a tableau to be optimal; no –ves in the objective row if it is a maximise problem and no +ves in the objective row if it is a minimise problem.
- Be prepared to explain why a particular tableau is or isn't optimal.

Exam tips

- Don’t forget to write the answer at the end. Too many people lose marks because they don’t interpret the solution.
- Make sure students are able to work out the value of slack variables in the tableau. These will tell you if there are any of the ‘raw materials’ left. This can be asked for in questions.
- Make sure they can read the values of all variables, including the slack variables form any completed tableau, even if it is not the final tableau.