Section 2: Trigonometric equations

Notes and Examples

In this section you learn how to solve trigonometric equations.

These notes contain subsections on

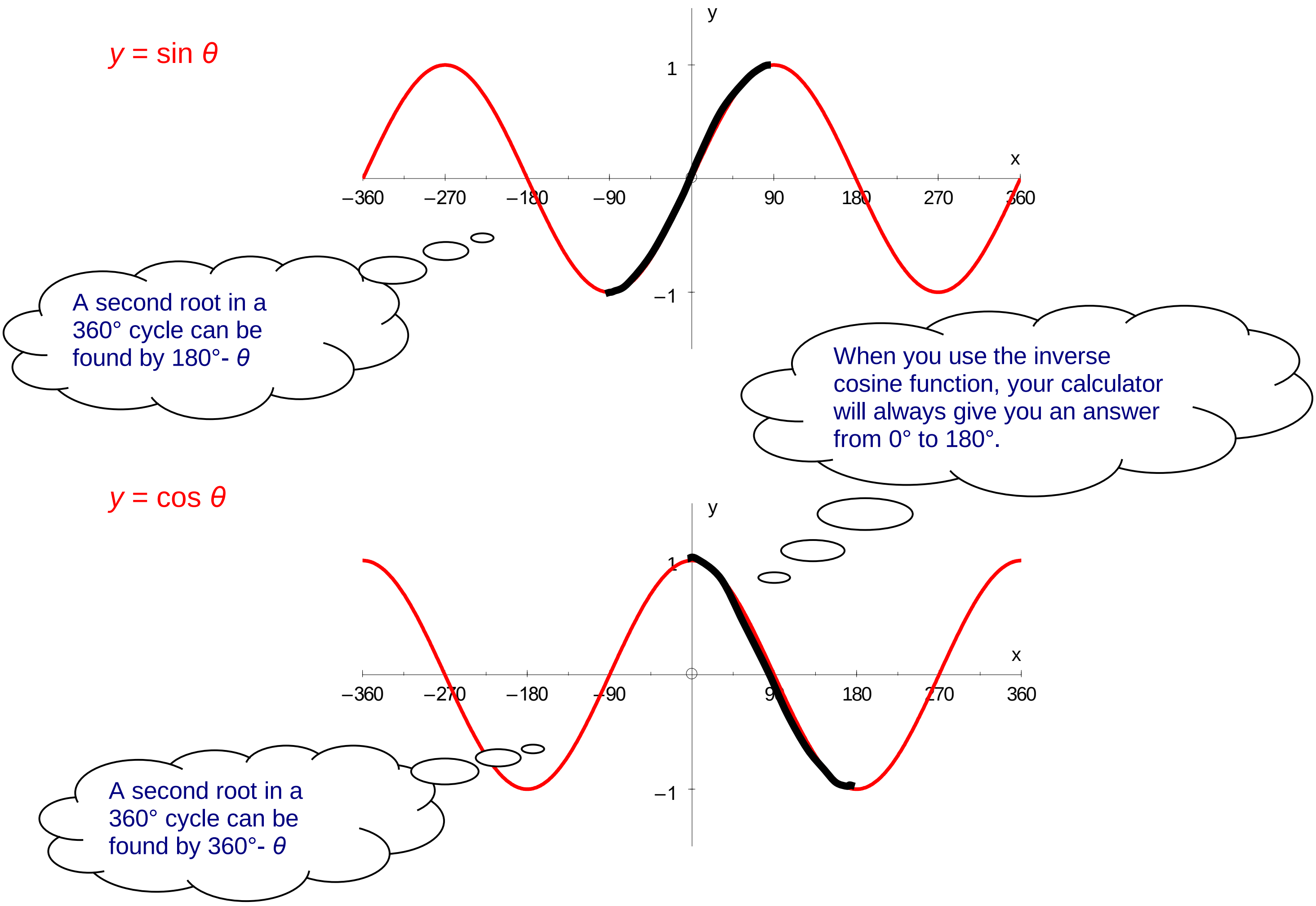
- **Principal values**
- **Solving simple trigonometric equations**
- **More complicated examples of trigonometric equations.**

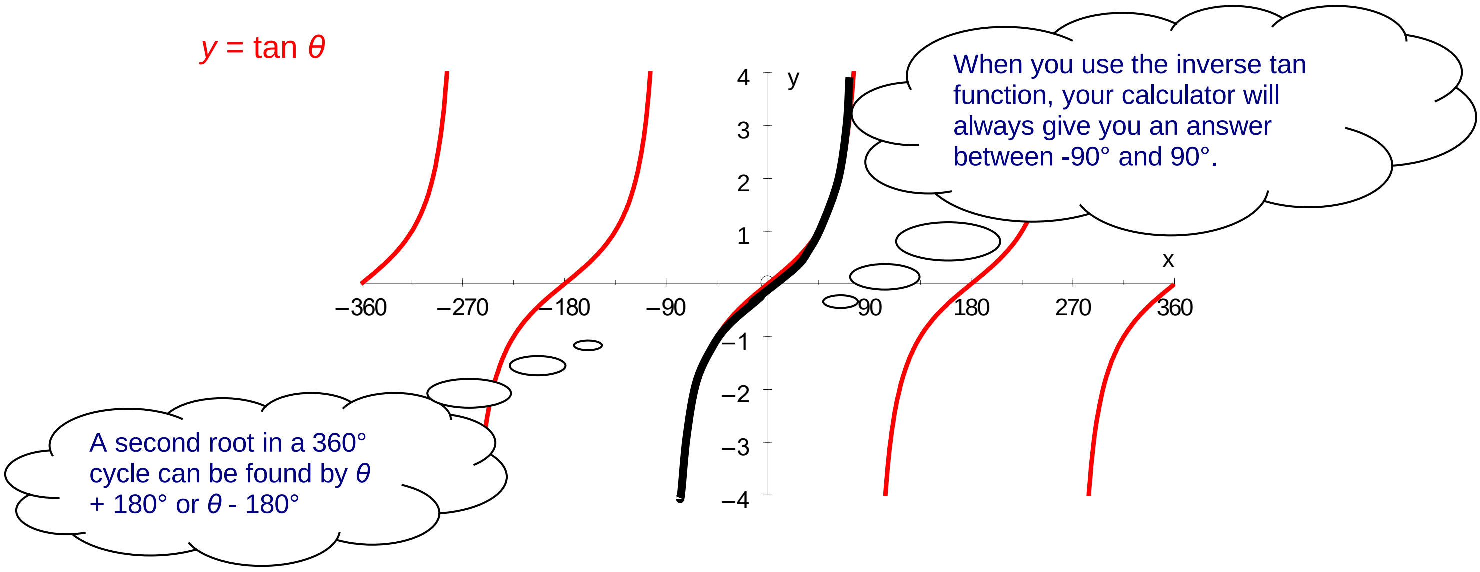
Principal values

There are infinitely many roots to an equation like $\sin\theta = \frac{1}{2}$.

Your calculator will only give one root – the ***principal value***.
You find this by pressing the calculator keys for arcsin 0.5 (or $\sin^{-1}$ 0.5 or invsin 0.5).
Check that you can get the answer of 30°.

You can find other roots by looking at the symmetry of the appropriate graph.





Alternatively, you can use the quadrant diagram to find other roots, by thinking about which quadrants the roots will be in.

Solving simple trigonometric equations

Because there are infinitely many roots to a trigonometric equation you are only ever asked to find a few of them! Any question at this level asking you to solve a trigonometric equation will also give you the interval or range of values in which the roots must lie, e.g. you might be asked to solve $\tan\theta = 2$ for $0° \le \theta \le 360°$.
You can only directly solve trigonometric equations like $\sin\theta = \frac{1}{2}$ or $\cos\theta = \frac{1}{4}$ or $\tan\theta = -2$. Here is an example.

Example 1

Solve $\sin\theta = \dfrac{\sqrt{3}}{2}$ for $-360° \le \theta \le 360°$.

Solution

$\sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = 60°$

There will be a second root in the second quadrant.
180° - 60° = 120° is also a root.
Since $y = \sin\theta$ has a period of 360° any other roots can be found by adding/subtracting 360° to these two roots.
So the other roots are:

60° - 360° = -300°
and 120° - 360° = -240°

So the values of θ for which $\sin\theta = \dfrac{\sqrt{3}}{2}$ are -300°, -240°, 60°, 120°.

Almost all equations of the type in Example 1 have two roots in the range $0° \le x < 360°$.

However, this is not true of all trigonometric equations. Suppose, for example, that you want to solve the equation $\sin 2x = 0.5$ in the range $0° \le x < 360°$. You can find that $2x = 30°$ or $150°$, which means that $x = 15°$ or $75°$. However, there are two further roots in the range $0° \le x < 360°$, given by $2x = 30° + 360° = 390° \Rightarrow x = 195°$, and $2x = 150° + 360° = 510° \Rightarrow x = 255°$. So there are four roots: $x = 15°, 75°, 195°, 255°$.

This means that if you are solving an equation of the form $\sin nx = k$, you need to adjust the range in which you look for initial roots.

Example 2

Solve each of the following equations in the given range:

(i) $\tan 3x = 1$ for $0 \le x < 360°$

(ii) $\cos(2x + 40°) = 0.5$ for $-180° < x \le 180°$

Solution

(i) $\tan 3x = 1$

$3x = 45°,\ 225°,\ 405°,\ 585°,\ 765°,\ 945°$

$x = 15°,\ 75°,\ 135°,\ 195°,\ 255°,\ 315°$

> You need to look for roots for $3x$ in the range 0° to 1080°

(ii) $\cos(2x + 40°) = 0.5$

$2x + 40° = -300°,\ -60°,\ 60°,\ 300°$

$2x = -340°,\ -100°,\ 20°,\ 260°$

$x = -170°,\ -50°,\ 10°,\ 130°$

> The lower end of the range you need to look in is $-180° \times 2 + 40° = -320°$, and the upper end is $180° \times 2 + 40° = 400°$

More complicated trigonometric equations

Any more complicated equations need to be manipulated algebraically before they can be solved. There are a number of techniques you can use:

1. Rearrange the equation to make $\cos\theta, \sin\theta$ or $\tan\theta$ the subject.
2. Check to see if the equation factorises to give two (or more) equations which involve just one trigonometric function (see Example 3).
   If it is a quadratic in either $\sin\theta$, $\cos\theta$, or $\tan\theta$ it can either be factorised or solved using the formula for solving quadratic equations (see Example 4).
3. If the equation involves just $\sin\theta$ and $\cos\theta$ (and no powers), check to see if you can use the identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ (see Example 5).
4. If the equation contains a mixture of trigonometric functions (e.g. $\cos^2\theta$ and $\sin\theta$) then you may need to use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to make it a quadratic in either $\sin\theta$, $\cos\theta$, or $\tan\theta$ (see Example 6).

Example 3

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Solve $2\cos\theta\sin\theta + \cos\theta = 0$ for $0° \le \theta \le 360°$.

Solution

$2\cos\theta\sin\theta + \cos\theta = 0$ can be factorised as there is $\cos\theta$ in both terms on the LHS.

Factorise: $\cos\theta(2\sin\theta + 1) = 0$

So either $\cos\theta = 0$ or $2\sin\theta + 1 = 0$

> It is wrong to divide through by $\cos\theta$ because you lose the roots to $\cos\theta = 0$.

$\cos\theta = 0 \Rightarrow \theta = 90°$

360° - 90° = 270° is also a root.

$2\sin\theta + 1 = 0 \Rightarrow \sin\theta = -\frac{1}{2}$

This has roots in the third and fourth quadrants.

The roots are 180° + 30° = 210° and 360° - 30° = 330°.

So the values of θ for which $2\cos\theta\sin\theta + \cos\theta = 0$ are 90°, 210°, 270° and 330°.

In Example 4 you need to solve a quadratic equation.

Example 4

Solve $2\cos^2\theta + 3\cos\theta = 2$ for $0° \le \theta \le 360°$.

Solution

$2\cos^2\theta + 3\cos\theta = 2$ is a quadratic equation in $\cos\theta$

Rearrange the quadratic: $2\cos^2\theta + 3\cos\theta - 2 = 0$

Let $\cos\theta = x$: $2x^2 + 3x - 2 = 0$

Factorise: $(2x - 1)(x + 2) = 0$

$x = \frac{1}{2}$ or $x = -2 \Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -2$

> You can replace $\cos\theta$ with x to make things simpler! Or factorise straightaway to get: $(2\cos\theta - 1)(\cos\theta + 2) = 0$ and then solve.

$\cos\theta = -2$ has no real roots.

So we need to solve $\cos\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = 60°$

There is also a root in the 4th quadrant, so 360° - 60° = 300° is also a root.

So the values of θ for which $2\cos^2\theta + 3\cos\theta = 2$ are 60° and 300°.

In the next example you need to use the identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$.

Example 5

Solve $\sin\theta - 2\cos\theta = 0$ for $0° \le \theta \le 360°$.

Solution

You need to rearrange the equation.

$\sin\theta - 2\cos\theta = 0$

Dividing by cos θ: $\dfrac{\sin\theta}{\cos\theta} - 2 = 0$

Since $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$: $\tan\theta - 2 = 0$

$\Rightarrow \tan\theta = 2$

$\Rightarrow \theta = 63.4°$ to 1 d.p.

You can safely divide by cos θ because it can't be equal to 0. If it were then sin θ would also have to be 0 and cos θ and sin θ are never both 0 for the same value of θ.

There is also a root in the 3rd quadrant.
So 63.4° + 180° = 243.4° is also a root.

So the values of θ for which $\sin\theta - 2\cos\theta = 0$ are 63.4° and 243.4° to 1 d.p.

In the next example you need to use the trigonometric identity $\sin^2\theta + \cos^2\theta \equiv 1$.

Example 6
Solve $\sin^2 x + \sin x = \cos^2 x$ for $0° \le x \le 360°$

Solution
Rearranging the identity gives: $\sin^2\theta + \cos^2\theta \equiv 1$
$\cos^2 x \equiv 1 - \sin^2 x$ ①

Substituting ① into the equation $\sin^2 x + \sin x = \cos^2 x$ gives:
$\sin^2 x + \sin x = 1 - \sin^2 x$

This is a quadratic in $\sin x$.
Rearranging: $2\sin^2 x + \sin x - 1 = 0$
Rearranging: $2\sin^2 x + \sin x - 1 = 0$
This factorises to give: $(2\sin x - 1)(\sin x + 1) = 0$

So either: $2\sin x - 1 = 0$ or $\sin x + 1 = 0$

$\Rightarrow \sin x = \frac{1}{2}$ $\Rightarrow \sin x = -1$

$\Rightarrow x = 30°$ or $150°$ $\Rightarrow x = 270°$

So the roots to $\sin^2 x + \sin x = \cos^2 x$ are $x = 30°, 150°$ or $270°$

