Graphs and transformations
Section 2: Transformations of graphs
Notes and Examples

These notes contain subsections on:

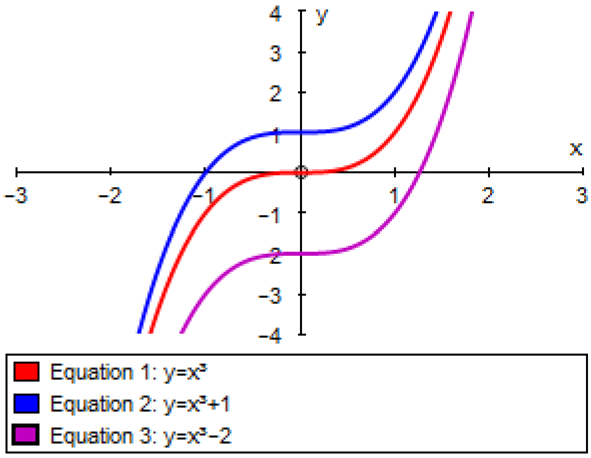
Vertical translations of the form y = f(x) + a

When the curve $y = f(x)$ is transformed onto the curve $y = f(x) + a$, for any particular value of x, a is added to the value of y. This has the effect of moving the whole curve a units upwards if a is positive, and a units downwards if a is negative. This is a translation of a units parallel to the y-axis, or, using vector notation, a translation of $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

In general:

> For any function f(x), the graph of $y = f(x) + a$ can be obtained from the graph of $y = f(x)$ by translating it through a units in the positive y direction.

The diagram below shows a graph $y = f(x)$ in red (in this case $f(x) = x^3$), the graph $y = f(x) + 1$ in blue, and the graph $y = f(x) - 2$ in purple.



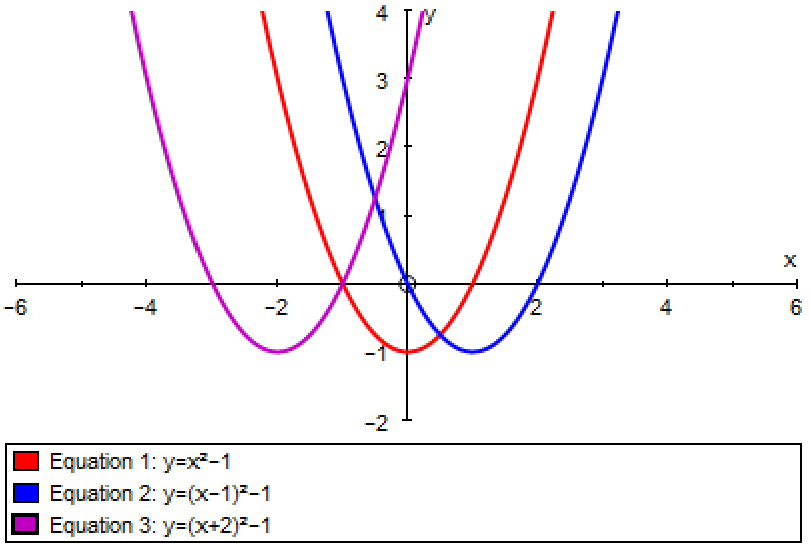
Horizontal translations of the form *y* = f(*x* – *a*)

When the curve $y = \mathrm{f}(x)$ is transformed into the curve $y = \mathrm{f}(x - a)$, for any particular value of y, the value of x must be a units greater to obtain the same value of y. This has the effect of moving the whole curve a units to the right if a is positive, and a units to the left if a is negative. This is a translation of a units parallel to the x-axis, or, using vector notation, a translation of $\begin{pmatrix} a \\ 0 \end{pmatrix}$.

In general:

> For any function f(x), the graph of $y = \mathrm{f}(x - a)$ can be obtained from the graph of $y = \mathrm{f}(x)$ by translating it through a units in the positive x direction.

The diagram below shows a graph $y = \mathrm{f}(x)$ in red (in this case $\mathrm{f}(x) = x^2 - 1$), the graph $y = \mathrm{f}(x - 1)$ in blue, and the graph $y = \mathrm{f}(x + 2)$ in purple.



Combined translations

Translating the graph $y = \mathrm{f}(x)$ by the vector $\begin{pmatrix} s \\ t \end{pmatrix}$ (i.e. s units to the right and t units vertically upwards) gives the graph $y = \mathrm{f}(x - s) + t$. This is simply a combination of the two translations already discussed.
In general:

> For any function f(x), the graph of $y = \mathrm{f}(x - s) + t$ can be obtained from the graph of $y = \mathrm{f}(x)$ by translating it through s units in the positive x direction and t units in the positive y direction.

You can rewrite this last result as:

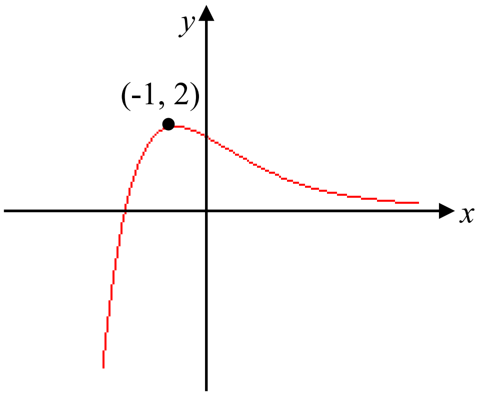
When the graph of $y = f(x)$ is translated through s units in the positive x direction and t units in the positive y direction, the resulting graph is given by $y - t = f(x - s)$.

This shows the symmetry of the results. Translating the graph through s units in the positive x direction is equivalent to replacing x with $x - s$. Translating the graph through t units in the positive y direction is equivalent to replacing y with $y - t$.

These ideas can be generalised to any graph.

Example 1

The diagram below shows the graph $y = f(x)$.



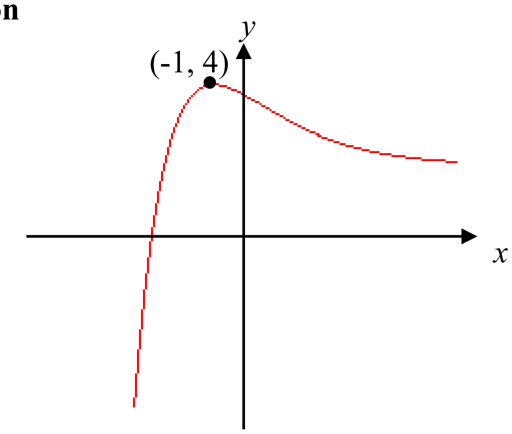
Sketch the graphs of :

(i) $y = f(x) + 2$
(ii) $y = f(x) - 1$
(iii) $y = f(x - 2)$
(iv) $y = f(x + 1)$
(v) $y = f(x - 1) - 2$

showing the coordinates of the turning point in each case.

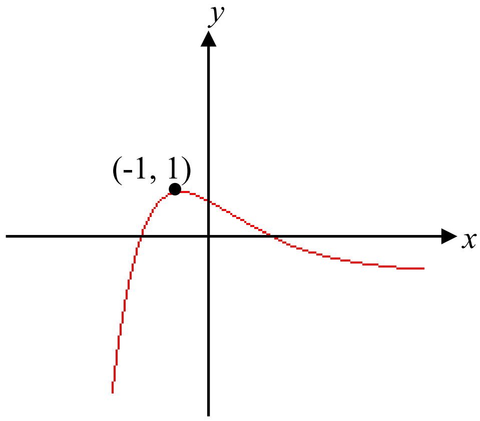
Solution

(i)



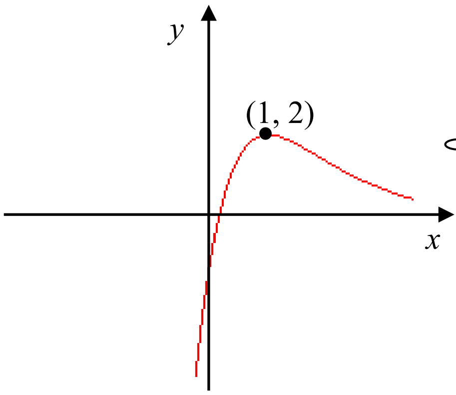
$y = f(x) + 2$ represents a translation of 2 units vertically upwards.

(ii)



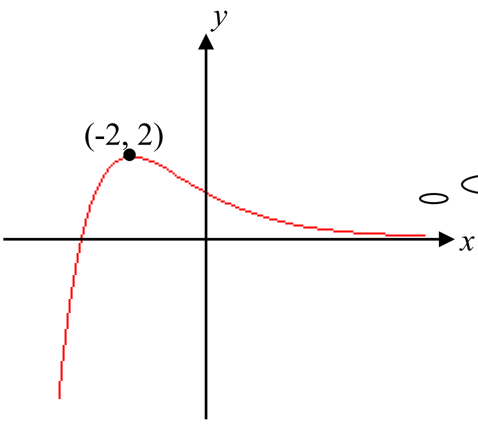
$y = f(x) - 1$ represents a translation of 1 unit vertically downwards.

(iii)



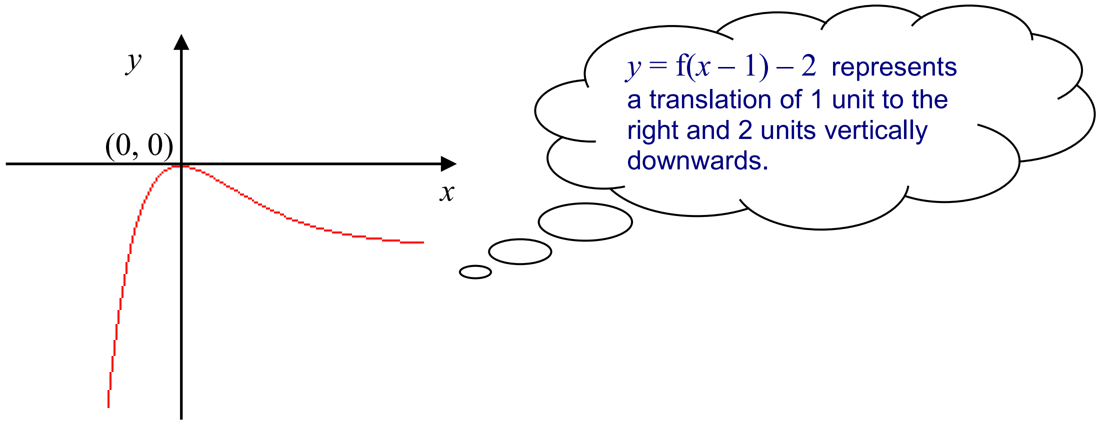
$y = f(x - 2)$ represents a translation of 2 units to the right.

(iv)



$y = f(x + 1)$ represents a translation of 1 unit to the left.

(v)



Vertical one-way stretches of the form *y* = *a*f(*x*)

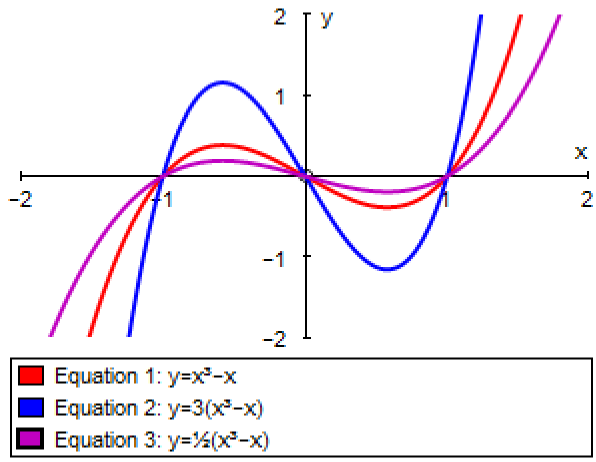
When the curve $y = \mathrm{f}(x)$ is transformed into the curve $y = a\mathrm{f}(x)$, for any particular value of x, the value of y is multiplied by a. This has the effect of stretching the curve by a scale factor of a in the y direction. (Of course, if a is less than 1, then the graph will be compressed rather than stretched).

In general:

> For any function f(x), and any positive value of a, the graph of $y = a\mathrm{f}(x)$ can be obtained from the graph of $y = \mathrm{f}(x)$ by a stretch of scale factor a parallel to the y-axis.

You can rewrite $y = a\mathrm{f}(x)$ as $\frac{y}{a} = \mathrm{f}(x)$. So replacing y with $\frac{y}{a}$ results in a stretch of scale factor a parallel to the y-axis.

The diagram below shows a graph $y = \mathrm{f}(x)$ in red (in this case $\mathrm{f}(x) = x^3 - x$), the graph $y = 3\mathrm{f}(x)$ in blue, and the graph $y = \frac{1}{2}\mathrm{f}(x)$ in purple.



Horizontal one-way stretches of the form $y = f(ax)$

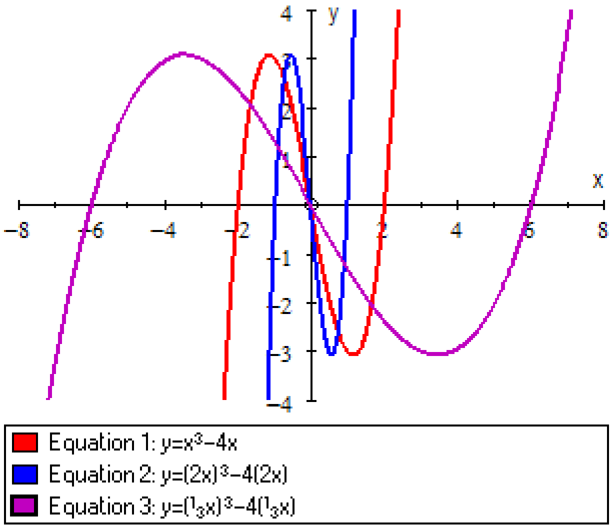
When the curve $y = \mathrm{f}(x)$ is transformed into the curve $y = \mathrm{f}(ax)$, for any particular value of y, the value of x must be multiplied by $\frac{1}{a}$ to obtain the same value of y. This has the effect of stretching the curve by a scale factor of $\frac{1}{a}$ in the x direction. (Of course, if a is greater than 1, then the graph will be compressed rather than stretched).

In general:

> For any function $\mathrm{f}(x)$, and any positive value of a, the graph of $y = \mathrm{f}(ax)$ can be obtained from the graph of $y = \mathrm{f}(x)$ by a stretch of scale factor $\frac{1}{a}$ parallel to the x-axis.

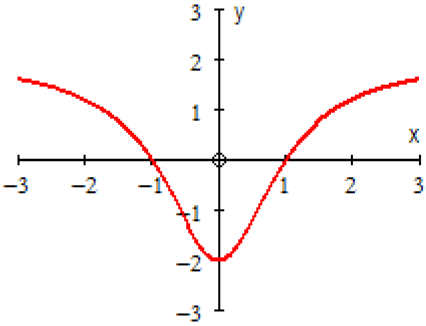
Similarly, the graph of $y = \mathrm{f}\left(\frac{x}{a}\right)$ is obtained from the graph of $y = \mathrm{f}(x)$ by a stretch of scale factor a parallel to the x-axis. So, just as replacing y with $\frac{y}{a}$ results in a stretch of scale factor a parallel to the y-axis, replacing x with $\frac{x}{a}$ results in a stretch of scale factor a parallel to the x-axis.

The diagram below shows a graph $y = \mathrm{f}(x)$ in red (in this case $\mathrm{f}(x) = x^3 - 4x$), the graph $y = \mathrm{f}(2x)$ in blue, and the graph $y = \mathrm{f}(\frac{1}{3}x)$ in purple.



Example 2

The diagram below shows the graph of $y = \mathrm{f}(x)$.



The graph cuts the x-axis at (1, 0) and (-1, 0) and the y-axis at (0, -2).

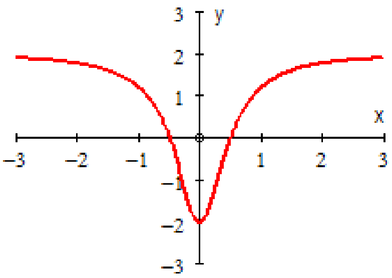
Sketch the graphs of:

(i) $y = \mathrm{f}(2x)$
(ii) $y = 3\mathrm{f}(x)$
(iii) $y = \mathrm{f}\left(\frac{1}{3}x\right)$
(iv) $y = \frac{1}{4}\mathrm{f}(x)$

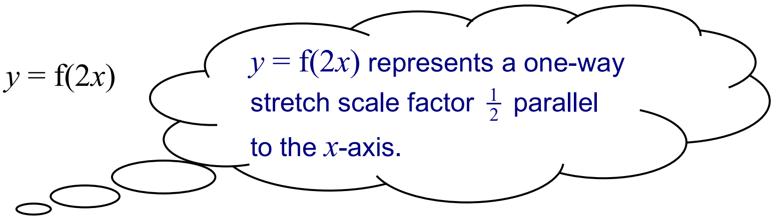
giving the coordinates of the points where the graph crosses the coordinate axes in each case.

Solution

(i)

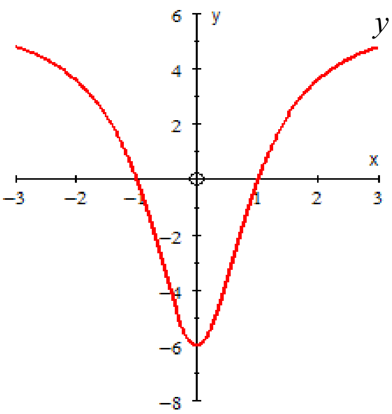


$y = \mathrm{f}(2x)$

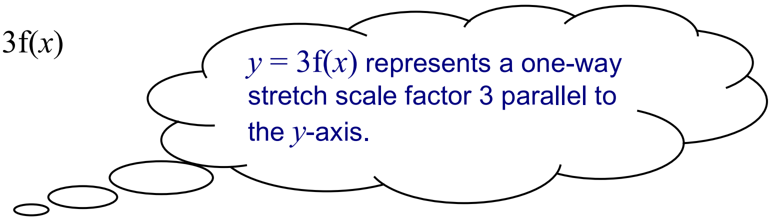


The graph cuts the x-axis at $\left(\frac{1}{2}, 0\right)$ and $\left(-\frac{1}{2}, 0\right)$ and the y-axis at (0, -2).

(ii)

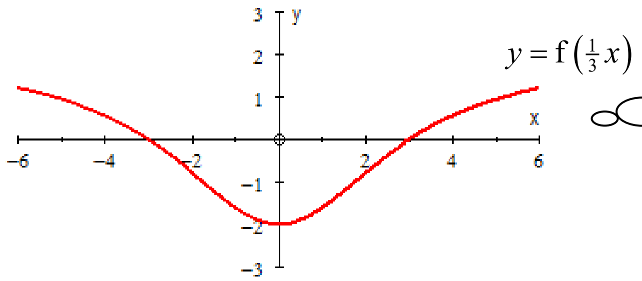


$y = 3\mathrm{f}(x)$



The graph cuts the x-axis at (1, 0) and (-1, 0) and the y-axis at (0, -6).

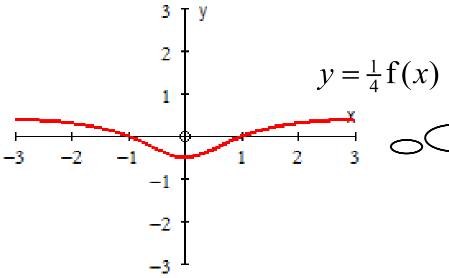
(iii)



$y = \mathrm{f}\left(\frac{1}{3}x\right)$ represents a one-way stretch scale factor 3 parallel to the x-axis.

The graph cuts the x-axis at (3, 0) and (-3, 0) and the y-axis at (0, -2).

(iv)



$y = \frac{1}{4}\mathrm{f}(x)$ represents a one-way stretch scale factor $\frac{1}{4}$ parallel to the y-axis.

The graph cuts the x-axis at (1, 0) and (-1, 0) and the y-axis at $\left(0, -\frac{1}{2}\right)$.

Reflections

In fact these are just special cases of the one-way stretches you already know about. The equation $y = -\mathrm{f}(x)$ represents a stretch with scale factor -1 parallel to the y axis, which is the same as reflection in the x axis. The equation $y = \mathrm{f}(-x)$ represents a stretch with scale factor -1 parallel to the x axis, which is the same as reflection in the y axis.

- $y = -\mathrm{f}(x)$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is reflected in the x axis
- $y = \mathrm{f}(-x)$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is reflected in the y axis

Example 3

Each of the following transformations is applied to the graph of the quadratic function $\mathrm{f}(x) = x^2 + 2x - 1$. Find the equation of the new curve in each case.

(i) Horizontal translation 3 units to the left

(ii) Stretch, scale factor 2, parallel to the y-axis

(iii) Reflection in the y-axis

Solution

(i) $y = \mathrm{f}(x+3)$

$$= (x+3)^2 + 2(x+3) - 1$$
$$= x^2 + 6x + 9 + 2x + 6 - 1$$
$$= x^2 + 8x + 8$$

(ii) $y = 2\mathrm{f}(x)$

$$= 2(x^2 + 2x - 1)$$
$$= 2x^2 + 4x - 2$$

(iii) $\mathrm{f}(-x) = (-x)^2 + 2(-x) - 1$

$$= x^2 - 2x - 1$$

Transformations of trigonometric graphs

All the rules above can be applied to trigonometric graphs.

Example 4

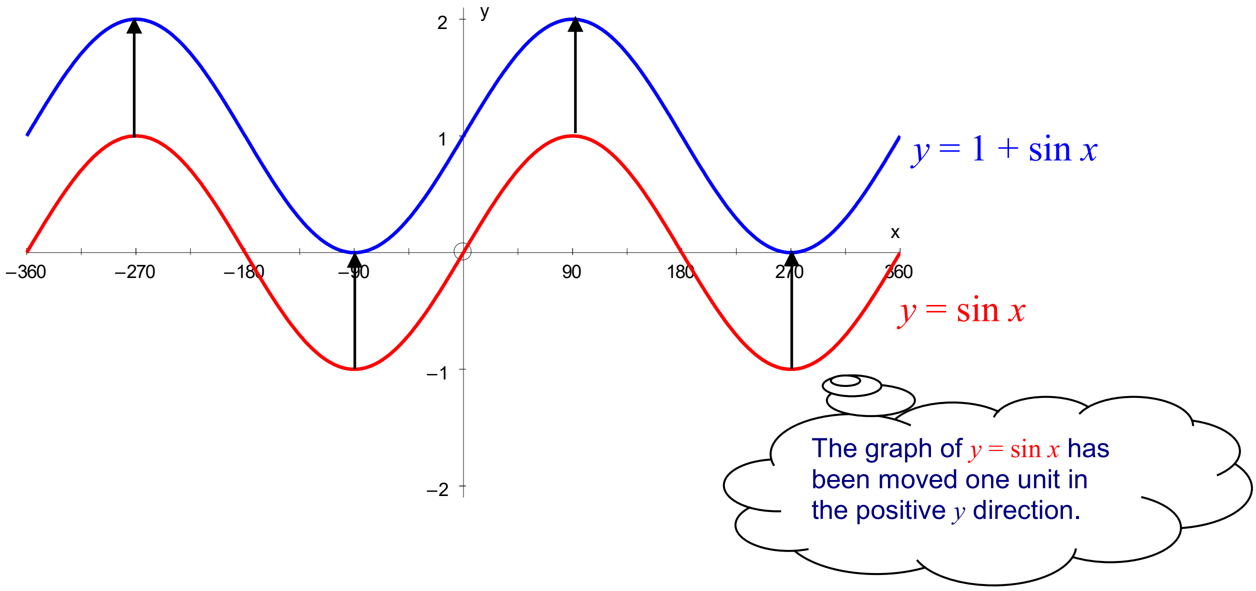
Sketch the graphs of

(i) $y = 1 + \sin x$

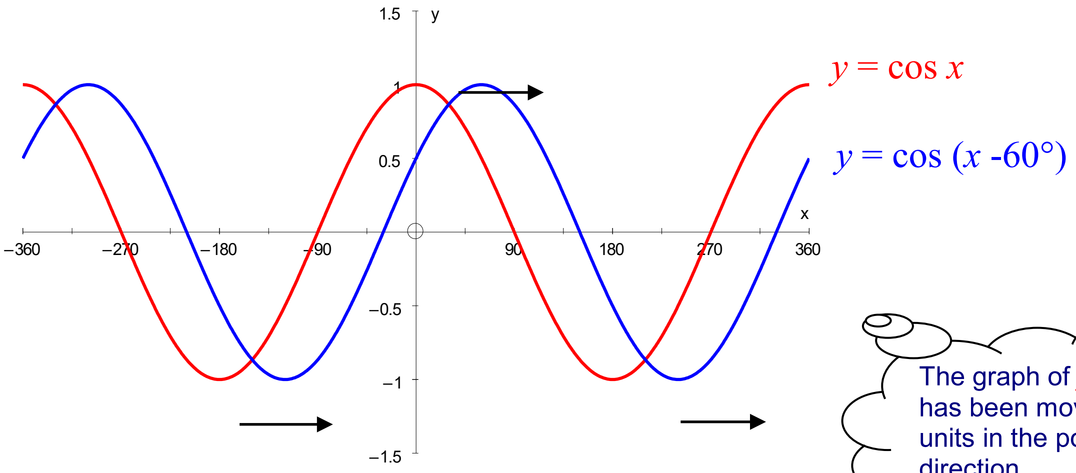
(ii) $y = \cos(x - 60°)$

Solution

(i) This is a translation of the graph $y = \sin x$ by 1 unit vertically upwards.



(ii) This is a translation of the graph $y = \cos x$ by 60° to the right.



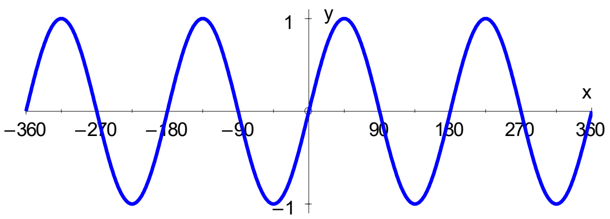
The graph of $y = \cos x$ has been moved 60 units in the positive x direction.

Horizontal stretches affect the **period** of a trigonometric graph. For example, the graph of $y = \cos 2x$ is obtained by stretching the graph of $y = \cos x$ by a factor of $\frac{1}{2}$ (i.e. compressing it). This graph therefore has a period half that of the period of the graph of $y = \cos x$.

Vertical stretches affect the amplitude of a trigonometric graph. For example, the graph of $y = 2\sin x$ is obtained by stretching the graph of $y = \sin x$ by a factor of 2. The period of this graph is the same as that of $y = \sin x$, but its amplitude is doubled.

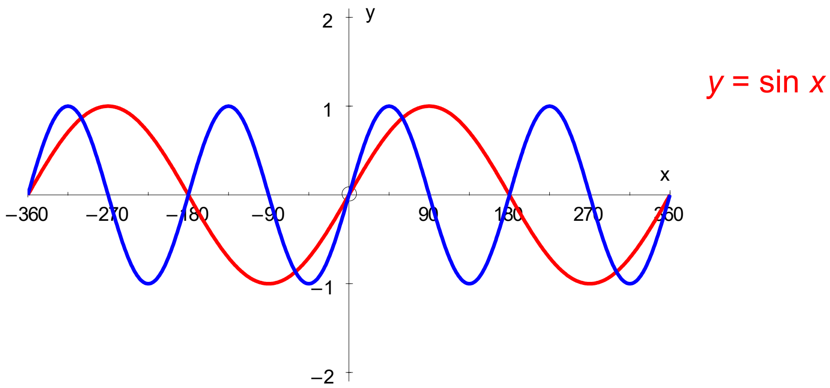
Example 4

Write down the equation of the following graph:



Solution

Compare the graph with that of $y = \sin x$:



The x co-ordinate of every point on the graph of $y = \sin x$ has been halved. Therefore the equation is $y = \sin 2x$.