Ritangle 2020, Questions and Answers

1-5; Preliminary Questions.

6-24; Main Contest Questions.

A-H; Final Task Questions.

+ the rest of the Final Question.

You may need to use a graphing program,

construct an Excel spreadsheet,

or research on the internet to solve some of these problems.

It's not necessary to solve every problem

to solve the whole puzzle (but it helps!)

Your answer each time will need to be converted into a final answer

by multiplying by one number, adding another

and taking the integer part of the result.

The integer part of x is found by throwing away the decimal part of x.

Thus the integer part of 4.73267 is 4, and the integer part of 5 is 5.

Notice that if x is not an integer,

you always round down here, and never up.

Make a note of your final answers for these first 24 questions,

because you will need them again later.

When you find a final answer to a question,

say 12345, then if you visit the appropriate webpage

that will confirm you have the right answer.

Please don't share your solutions with anyone outside your team.

To be in with a chance of winning,

you must hold on to your working for the Final Question, Stages 1, 2 and 3.

Question 1 (Monday 5th October)

Kanesha is examining her gas bill. She sees that a unit of gas sells for £a. This price is then increased by b%. A month later, Kanesha notices that the new price is reduced by (b - 1)% , and that takes the unit price back to £a. What (to 3s.f.) is b?

To get your final answer, multiply your value by 79155, add 10 and take the integer part.

Solution:

$$a\left(1+\frac{b}{100}\right)-a\left(1+\frac{b}{100}\right)\frac{b-1}{100}=a.$$

Dividing by a, multiplying out and simplifying gives

$$b^2-b-100=0\Rightarrow b=10.5(3s.f.)$$

Final answer = 831137.

Question 2 (Monday 12th October)

Evan rolls a fair dice 10 times.

What is P(exactly four fours)/P(exactly five fives)?

To get your final answer, multiply your value by 9057, add 10 and take the integer part.

Solution:

P(exactly four fours) = $^{10}C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^6$.

P(exactly five fives) = $^{10}C_5\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^5$.

On dividing we get $\frac{^{10}C_4}{^{10}C_5}\times 5 = 5\times\frac{10!5!5!}{6!4!10!} = \frac{25}{6}$.

Final answer = 37747.

Question 3 (Monday 19th October)

Yu Yan is watching two cricket teams play a 20 overs a side game. The team chasing have been set a target of T runs, but Yu Yan can't see what T is on the scoreboard. After 10 overs they have scored 78 runs, and after 11 overs they have scored 88 runs. The run-rate (average runs already scored per over already bowled) and the required run-rate (required average runs per over remaining to be bowled) are calculated after the 10th and 11th overs, and Yu Yan notices that on the scoreboard they both increase by the same amount. What is T?

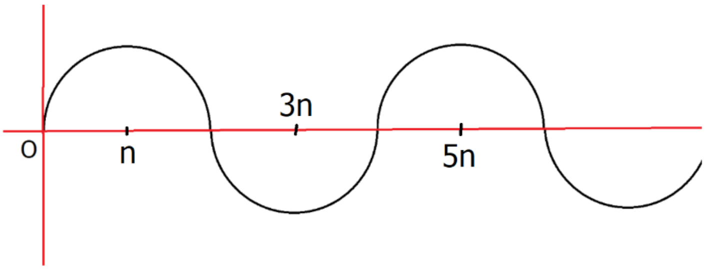
To get your final answer, multiply your value by 2199.7, add 6 and take the integer part.

Solution:

$$\frac{T-88}{9}=\frac{T-78}{10}+(8-7.8)\Rightarrow T=196.$$

Final answer = 431147.

Question 4 (Monday 2nd November)



Wendy defines the graph of a periodic function $f_n(x)$ as follows for positive x;

- a semicircle above the x-axis with centre (n, 0),
- a semicircle below the x-axis with centre (3n, 0),
- a semicircle above the x-axis with centre (5n, 0)....

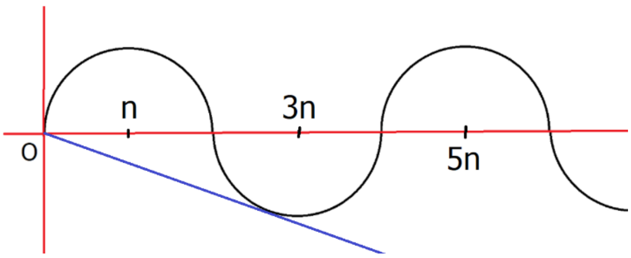
and so on, all with radius n.

Wendy discovers that if all the curves $y = f_n(x)$ are drawn for all positive integers n, then the line y = -mx (where m is finite and as large as possible) is a tangent to every curve. What is the value of m to 3s.f.?

To get your final answer, multiply your value by 123053, add 1 and take the integer part.

Solution:

Suppose that y = -mx is a tangent to the second semicircle for $y = f_n(x)$ (m cannot be larger than this for this curve).



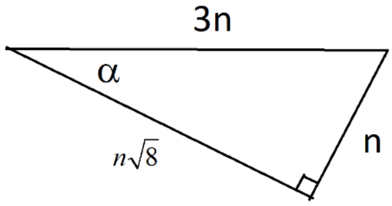
Method 1; So the semicircle is $(x-3n)^2 + y^2 = n^2$ and the line is $y = -mx.$ If the line is tangent to the semicircle, then the quadratic equation $(x-3n)^2 + (-mx)^2 = n^2$ will have equal roots.

This is $(m^2+1)x^2 - 6nx + 8n^2 = 0$, so $b^2 = 4ac$ gives us

$$36n^2 = 4(m^2+1)8n^2 \Rightarrow m = \frac{1}{2\sqrt{2}}$$, which is independent of n. Thus

$m = \frac{1}{2\sqrt{2}} = 0.354(3s.f.)$, and the line y = -mx is tangent to every curve.

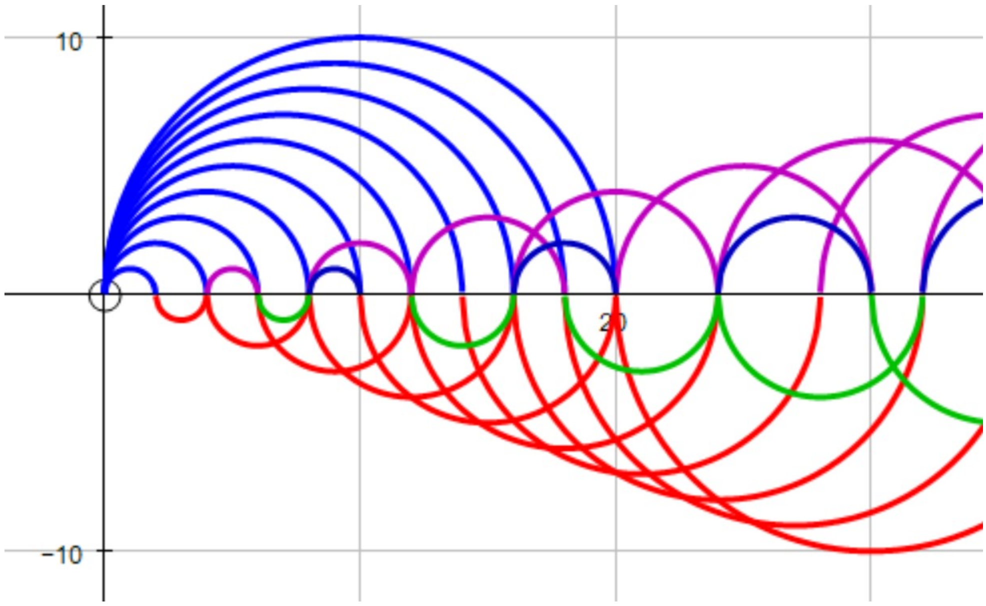
Method 2



$\tan\alpha = \frac{1}{\sqrt{8}}$ which is independent of n (or $\sin\alpha = \frac{1}{3}$). Thus m = 0.354(3s.f.) as before.

Here are the curves y = $f_n(x)$ for n = 1 to 10.

With thanks to Jason Davies, https://www.jasondavies.com/primos/



Final answer = 43561.

Question 5 (Monday 9th November)

Olivia draws a quadrilateral with four interior angles that are in geometric progression. The common ratio r is an integer. The smallest of the four angles is α degrees, and 10α is an integer. If Olivia has drawn the quadrilateral where r takes the largest possible value, what is α?

An Excel spreadsheet may be useful to you here.

To get your final answer, multiply your value by 2812, add 1 and take the integer part.

Solution:

We have that $\alpha + r\alpha + r^2\alpha + r^3\alpha = 360$, where r is the common ratio.

Thus $10\alpha = \dfrac{3600}{1+r+r^2+r^3}$ which is an integer.

1	900
2	240
3	90
4	42.35294118
5	23.07692308
6	13.8996139
7	9
8	6.153846154
9	4.390243902
10	3.240324032
11	2.459016393
12	1.909814324
13	1.512605042
14	1.218274112
15	0.995575221

A spreadsheet with r in the first column and 10α in the second shows that r has a maximum value of 7, giving $\alpha = 0.9$.

Final answer = 2531.

Question 6 (Wed 11th November)

Ramesh constructs an acute angle of size x degrees. He notices that

$\sin x^o$, $\cos x^o$ and $\tan x^o$ in that order are consecutive terms from an arithmetic progression. What is x to 3 s.f.?

A graphing program may be needed here to solve an equation.

To get your final answer, multiply your value by 5743, add 3 and take the integer part.

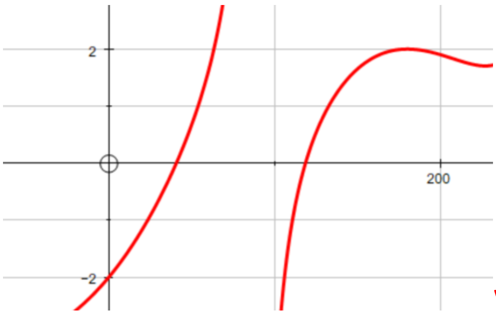
Solution:

We have $\sin x^o = a$, $\cos x^o = a + d$, $\tan x^o = a + 2d$.

This gives us $\tan x^o = \sin x^o + 2(\cos x^o - \sin x^o) \Rightarrow \tan x^o + \sin x^o - 2\cos x^o = 0$.

Thus .

Drawing the graph $y = \tan x^o + \sin x^o - 2\cos x^o$,



we can see on zooming in that the only possible value for x is 40.8 (3 s.f.).

Final answer = 234317.

Question 7 (Thursday 12th November)

Davina is examining the world record times for the marathon in various years. She writes down two such times as follows;

Belayneh Densamo (Ethiopia)	2:06:50	Rotterdam, 1988
Eliud Kipchoge (Kenya)	2:01:39	Berlin, 2018

Davina assumes the graph of world record time against year is linear, and on the basis of these two times calculates the expected world record time for the marathon in 2050 to the nearest second. What is this time?

Assume that all these marathons take place at the same time of year.

Give your answer as HMMSS.

To get your final answer, multiply your value by 4.33, add 134 and take the integer part.

Solution:

Method 1

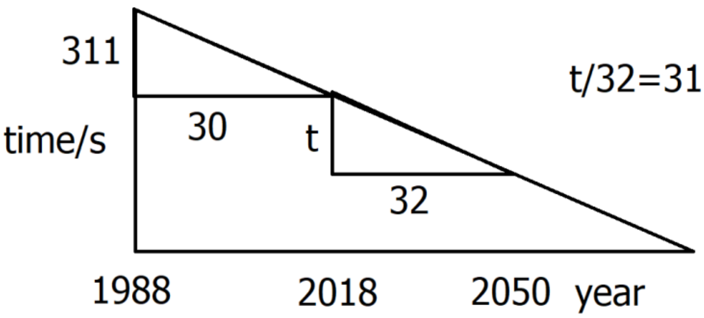
BD runs the marathon in 7610 secs.

EK runs the marathon in 7299 secs.

So the equation of the line we seek is $\frac{y-7299}{x-2018}=\frac{7610-7299}{1988-2018}$.

Putting in x = 2050 gives y = 1 hour 56 mins 7 secs, or 15607.

Method 2



$t/32=311/30$

Final answer = 67712.

Question 8 (Friday 13th November)

Sita puts 26 tiles bearing the letters A to Z into a bag, and draws three tiles one by one at random without replacing them as she goes.

What's the probability (to 3s.f.) that she draws the three tiles in alphabetical order?

To get your final answer, multiply your value by 1024832, add 1 and take the integer part.

Solution:

Method 1

For any choice of three letters, there are six equally likely possible orders, of which only one is alphabetical.

So the answer is 1/6 = 0.167(3s.f.)

Method 2

Using Excel;

		S	e	c	o	n	d		t	i	l	e																
		a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	
F	a		24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	300
i	b			23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	276
r	c				22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	253
s	d					21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	231
t	e						20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	210
	f							19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	190
t	g								18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	171
i	h									17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	153
l	i										16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	136
e	j											15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	120
	k												14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	105
	l													13	12	11	10	9	8	7	6	5	4	3	2	1	0	91
	m														12	11	10	9	8	7	6	5	4	3	2	1	0	78
	n															11	10	9	8	7	6	5	4	3	2	1	0	66
	o																10	9	8	7	6	5	4	3	2	1	0	55
	p																	9	8	7	6	5	4	3	2	1	0	45
	q																		8	7	6	5	4	3	2	1	0	36
	r																			7	6	5	4	3	2	1	0	28
	s																				6	5	4	3	2	1	0	21
	t																					5	4	3	2	1	0	15
	u																						4	3	2	1	0	10
	v																							3	2	1	0	6
	w																								2	1	0	3
	x																									1	0	1
	y																										0	0
	z		24	46	44	84	100	114	126	136	144	150	154	156	156	154	150	144	136	126	114	100	84	66	46	24	0	2600

$$\frac{2600}{26 \times 25 \times 24} = \frac{1}{6} = 0.167(3s.f.)$$

Final answer = 171147.

Question 9 (Monday 16th November)

Donald draws ten mathematical objects

that can be labelled from 3 to 12.

Donald puts them into a natural sequence as follows;

10-12-11-7-6-a-b-c-d-e,

where a, b, c, d and e are the digits 3, 4, 5, 8 and 9 in some order.

What is the five-digit number abcde?

To get your final answer, multiply your value by 0.0437, add 5 and take the integer part.

Solution:

The ten objects are

Triangle, Quadrilateral, Pentagon, Hexagon, Heptagon, Octagon, Nonagon, Decagon, Hendecagon, Dodecagon.

The natural sequence arises

when you put these into alphabetical order.

This gives 10-12-11-7-6-9-8-5-4-3, so abcde is 98543.

Final answer = 4311.

Question 10 (Tuesday 17^{th} November)

Ian constructs a large cube in layers from unit cubes as follows;

The first layer is one cube scoring 100. This layer scores 100.

The second layer is 26 cubes surrounding the first layer,

creating a solid 3x3x3 cube, with each of these new cubes scoring 99.

This new layer scores 26 x 99 = 2574.

The third layer is 98 cubes surrounding the second layer,

creating a solid 5x5x5 cube, with each of these cubes scoring 98.

This layer scores 98 x 98 = 9604.

This continues until Ian adds the 100th layer,

with each of these cubes scoring 1.

What is the score for the highest scoring layer?

To get your final answer, multiply your value by 0.1222, add 319 and take the integer part.

Solution:

Method 1

Layer	Number of cubes	Each score	Total score
1	1	100	100
2	$26 = 3^3-1^3$	99	2574
3	$98 = 5^3-3^3$	98	9604
…	…	…	…
n	$(2n-1)^3-(2n-3)^3$	101-n	s
	$24n^2-48n+26$		

So $s = (24n^2-48n+26)(101-n) \Rightarrow s = -24n^3+2472n^2-4874n+2626$.

So $ds/dn = -72n^2+4944n-4874$,

and putting this equal to 0 gives $n \approx 67.7$.

This suggests the highest score will happen when n = 67 or 68.

s(66) = 3549070, s(67) = 3554564, s(68) = 3555354, s(69) = 3551296.

Thus n = 68 gives us the largest value for s,

and the answer is 3555354.

Method 2

Graph $y = -24x^3+2472x^2-4874x+2626$, and find the maximum point.

Final answer = 434783.

Question 11 (Wednesday 18th November)

Oscar draws a pair of circles.

The circumferences of the two circles add to πa,

while their areas add to πb.

If a is 24, what is

(maximum possible value of b) – (minimum possible value of b)?

A point can be seen as a circle of zero radius.

To get your final answer, multiply your value by 15.1, add 56 and take the integer part.

Solution:

$$2\pi r + 2\pi R = 24\pi \Rightarrow r = 12 - R$$

$$\pi r^2 + \pi R^2 = \pi b \Rightarrow (12-R)^2 + R^2 - b = 0 \qquad *$$

$$\Rightarrow 2R^2 - 24R + 144 - b = 0$$

$$\Rightarrow R = \frac{24 \pm \sqrt{24^2 - 8(144-b)}}{4} = 6 \pm \frac{\sqrt{8b-576}}{4} \Rightarrow r = 6 \mp \frac{\sqrt{8b-576}}{4}.$$

So 8b $\geq$ 576 $\Rightarrow$ b $\geq$ 72 and $6 \geq \frac{\sqrt{8b-576}}{4} \Rightarrow b \leq 144$.

Thus 144 - 72 = 72.

Method 2

Completing the square on equation *, b = 2((R-6)^2+36).

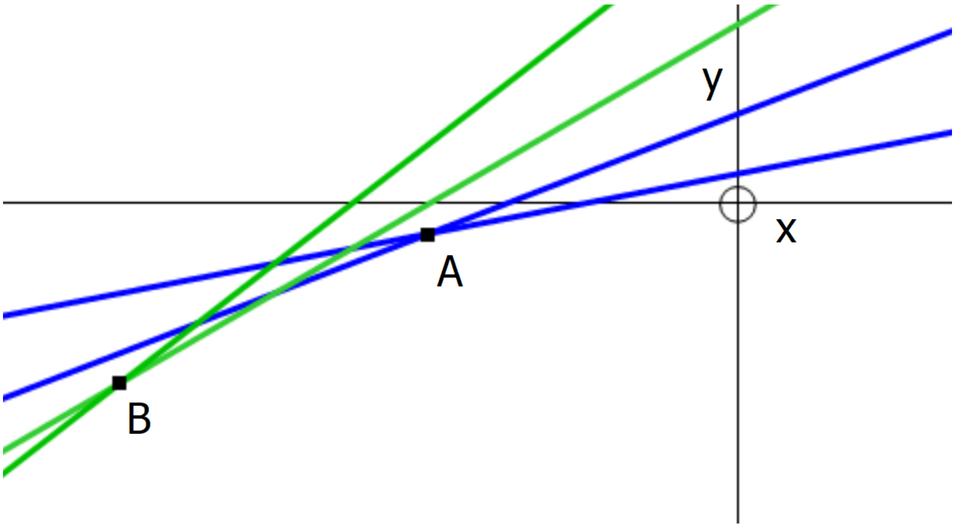
So the minimum value of b occurs when R = 6, when b = 72,

and the maximum value of b happens when R = 0 or 12,

when b = 144.

Final answer = 1143.

Question 12 (Thursday 19th November)



Paula draws the lines y = ax + 1 and y = 2ax + 3,

which meet at A.

She then draws the lines y = 3ax + 6 and 4ax + 10,

which meet at B.

The value a is a positive integer.

If AB = $\frac{\sqrt{629}}{5}$, what is a?

To get your final answer, multiply your value by 14223.1, add 1 and take the integer part.

Solution:

Solving y = ax + 1 and y = 2ax + 3 together simultaneously gives A = $\left(-\frac{2}{a},-1\right)$.

Solving y = 3ax + 6 and y = 4ax + 10 together simultaneously gives B = $\left(-\frac{4}{a},-6\right)$

.

$$\frac{629}{25}=AB^2=\left(-\frac{2}{a}-\left(-\frac{4}{a}\right)\right)^2+\left(-1--6\right)^2=25+\frac{4}{a^2}\Rightarrow a=5.$$

Final answer = 71116.

Question 13 (Friday 20th November)

Haris is making models of a regular tetrahedron T,

a regular octahedron O and a cube C.

He is using sticks of various lengths to make the models.

Each of the edges of the cube C is a 10cm long stick.

Haris wants the combined surface areas of T and O

to add to the total surface area of C.

He also wants the combined total stick-length for T and O

to add to the total stick-length needed for C.

Find the length of one of the sticks needed to build T in cm to 3s.f.

To get your final answer, multiply your value by 91, add 19 and take the integer part.

Solution:

Let the side-length of T be x cm and the side-length of O be y cm.

Considering the edges, 6x + 12y = 120 $\Rightarrow$x = 20-2y.

The area of one of T's faces is $\frac{\sqrt{3}x^2}{4}$ while the area of one of O's faces is $\frac{\sqrt{3}y^2}{4}$.

Thus $4\frac{\sqrt{3}x^2}{4}+8\frac{\sqrt{3}y^2}{4}=6\times 100$.

Substituting in for x, we get a quadratic equation for y

that solves to give

y = 12.6 (impossible) and 0.7074, and so x = 18.6 (3s.f.).

Final answer = 1711.

Question 14 (Monday 23rd November)

Aisha is trying to save £x.

After a week, she's saved a total of £p,

which is q% of the total required.

After a further week, she's saved a total of £q,

which is (p + 7.5)% of the total required.

Aisha notices that $p+q=\dfrac{x}{10}$.

What is x?

To get your final answer, multiply your value by 3694.1, add 15 and take the integer part.

Solution:

$$p=\frac{qx}{100},q=\frac{(p+7.5)x}{100},p+q=\frac{x}{10}\Rightarrow\frac{qx}{100}+q=x\Rightarrow q=\frac{100x}{x+100},p=\frac{x^2}{x+100.}$$

$$\Rightarrow\frac{100x}{x+100}=\frac{\left(\dfrac{x^2}{x+100.}+7.5\right)x}{100}\Rightarrow x=25,p=0.5,q=2.$$

So the answer is 25.

Final answer = 92367.

Question 15 (Tuesday 24^{th} November)

Nigel is initially baffled by this cryptic clue;

Neither impossible nor easy,

this elegant examination needs terrific insight

(made eventually straightforward, thankfully)

when everything looks very enigmatic…

But eventually he solves it to get a three digit number.

What is this number?

To get your final answer, multiply your value by 36.1, add 117 and take the integer part.

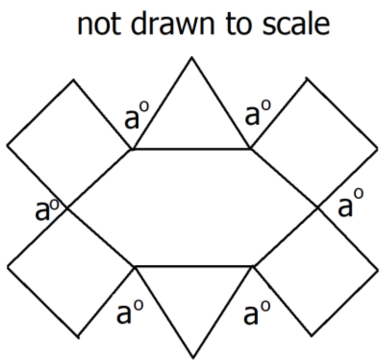
Solution:

This is an acrostic.

Nineteen times twelve = 228.

Final answer = 8347.

Question 16 (Wed 25th November)



Tom constructs the diagram above,

which consists of four squares and two equilateral triangles,

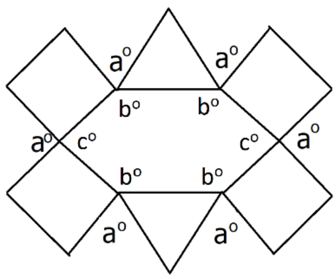
all of side 1 unit, that surround a hexagon.

The six angles labelled a° are all equal. Find a.

To get your final answer, multiply your value by 771.11, add 35 and take the integer part.

Solution:

Method 1



a + b + 150 = 360, a + c + 180 = 360, 4b + 2c = 720.

Thus a + b = 210, a + c = 180, 2b + c = 360.

Subtracting the second from the third, 2b – a = 180,

and adding the first gives 3b = 390.

Thus b = 130, a = 80, c = 100. So the answer is a = 80.

Method 2

$720 = 4b + 2c = 4(210 - a) + 2(180 - a) \Rightarrow a = 80.$

Final answer = 61723.

Question 17 (Thursday 26th November)

Una is examining the polynomial f(x) = ax^3 + bx^2 + cx + d,

where a, b, c and d are all integers.

She notices that $\frac{d^2(f(x)+x^3)}{dx^2} = 12\sqrt[3]{f(x)+3x^2}$ for all values of x.

What is a×b×c×d?

To get your final answer, multiply your value by 242.1, add 14 and take the integer part.

Solution:

Method 1

$$\frac{d(f(x))}{dx} = 3(a+1)x^2 + 2bx + c, \frac{d^2(f(x))}{dx^2} = 6(a+1)x + 2b.$$

$\Rightarrow (6(a+1)x + 2b)^3 - 12^3(ax^3 + bx^2 + cx + d + 3x^2) = 0$ for all values of x.

Multiplying out and collecting together like terms gives

$$216x^3(a^3 + 3a^2 - 5a + 1) + x^2(216a^2b + 432ab - 1512b - 5184) + 72x(ab^2 + b^2 - 24c) + 8(b^3 - 216d) = 0$$

.This means that the coefficient of each power of x here

must be zero (since this identity holds for all x).

$$a^3 + 3a^2 - 5a + 1 = 0 \Rightarrow (a-1)(a^2+4a-1) = 0 \Rightarrow a = 1 \text{ or } -2 \pm \sqrt{5} \Rightarrow a = 1.$$

Substituting in we have

$$-x^2(864b + 5184) + 144x(b^2 - 12c) + 8(b^3 - 216d) = 0$$

And so b = -6, c = 3, d = -1.

So the answer is 18.

Method 2

Define $h(x) = \left(\frac{6(a+1)x + 2b}{12}\right)^3 = \frac{(3x(a+1)+b)^3}{216} = ax^3 + (b+3)x^2 + cx + d$.

$$h(0) = \frac{b^3}{216} = d \quad .$$

Differentiating, $h'(x) = \frac{(a+1)(3x(a+1)+b)^2}{24} = 3ax^2 + 2(b+3)x + c \Rightarrow h'(0) = \frac{(a+1)b^2}{24} = c$.

Differentiating again,

$$h''(x) = (a+1)^2 \frac{3x(a+1)+b}{4} = 6ax + 2(b+3) \Rightarrow h''(0) = (a+1)^2 \frac{b}{4} = 2(b+3)$$ Differentiating again,

$$h'''(x) = \frac{3}{4}(a+1)^3 = 6a \Rightarrow a = 1, b = -6, c = 3, d = -1.$$

Final answer = 4371.

Question 18 (Friday 27th November)

Susanna cracks the code

1968119625 256144441361 368136400625-36181576

to get a two digit number. What is this number?

To get your final answer, multiply your value by 73.1, add 46 and take the integer part.

Solution:

A	B	C	D	E	F	G	H	I
1	4	9	16	25	36	49	64	81
J	K	L	M	N	O	P	Q	R
100	121	144	169	196	225	256	289	324
S	T	U	V	W	X	Y	Z	
361	400	441	484	529	576	625	676	

Decoding we have NINE PLUS FIFTY-SIX = 65.

Final answer = 4797.

Question 19 (Monday 30th November)

Francois is making models of a cube and a dodecahedron

using sticks for the edges.

He wants each solid to have the same surface area.

If the cube has surface area 100cm^2,

then what in cm (to 3s.f.) is the total stick length required

to make the pair of models?

To get your final answer, multiply your value by 414.9, add 23 and take the integer part.

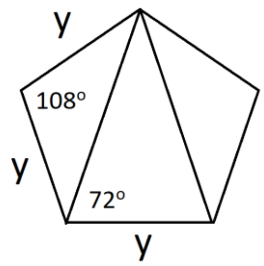
Solution:

Let the side of the cube be x cm

and the side of the dodecahedron be y cm.

We need 12x + 30y.

$6x^2 = 100$, so $x = \sqrt{\frac{50}{3}} = 4.082483...$



$$100 = 12\left(2\frac{1}{2}y^2 \sin 108^o + \frac{1}{2}y\frac{y}{2}\tan 72^o\right) \Rightarrow y = 2.200822......$$

Thus 12x + 30y = 115 cm (3s.f.)

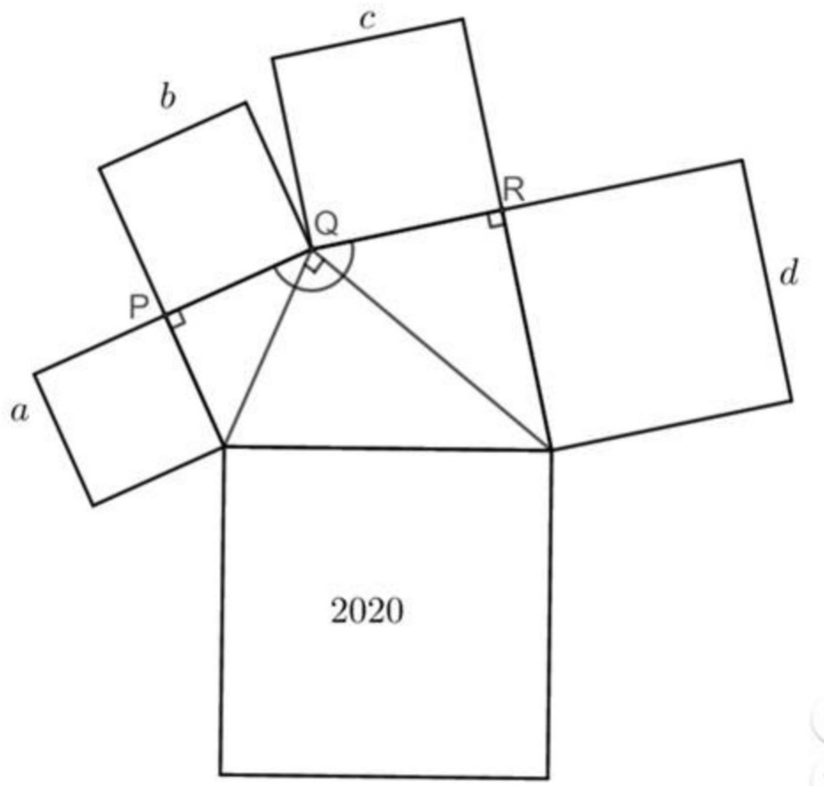
Final answer = 47736.

Question 20 (Tuesday 1st December)

Eleanor is reading about mathematics on the internet.

She discovers that $17^2+19^2+23^2+29^2 = 2020$,

so 2020 is the sum of the squares of four consecutive primes.



The diagram shows this fact,

via five squares surrounding three right-angled triangles.

The sides a, b, c, d are 17, 19, 23, 29 in some order.

In the case when a < b, Eleanor wonders what order

would bring the marked angle PQR

as close as possible to 180° without exceeding it.

Give your answer as the values for a, b, c and d concatenated.

An Excel spreadsheet may be useful to you here.

To get your final answer, multiply your value by 0.000079, add 8 and take the integer part.

Solution:

We need $\arctan(a/b) + \arctan(d/c) \approx 90^{o}$.

We could put the 12 possibilities into a spreadsheet (using radians).

a	b	c	d	arctan(a/b)	arctan(d/c)	sum	pi/2-sum
19	23	17	29	0.690446457	1.040580554	1.731027	-0.160231011
17	19	23	29	0.729899658	0.900274769	1.630174	-0.059378427
17	23	19	29	0.636508216	0.990793553	1.627302	-0.056505768
19	29	17	23	0.580002774	0.934288111	1.514291	0.056505115
23	29	17	19	0.670521558	0.840896669	1.511418	0.059377773
17	29	19	23	0.530215773	0.88034987	1.410566	0.160230357
23	29	19	17	0.670521558	0.729899658	1.400421	0.170374784
17	19	29	23	0.729899658	0.670521558	1.400421	0.170374784
17	29	23	19	0.530215773	0.690446457	1.220662	0.35013377
19	23	29	17	0.690446457	0.530215773	1.220662	0.35013377
19	29	23	17	0.580002774	0.636508216	1.216511	0.35428501
17	23	29	19	0.636508216	0.580002774	1.216511	0.35428501

Now ordering by the final column, we need the smallest positive result.

arctan(19/29) + arctan(23/17) is the best we can do.

Notice that (ignoring rounding errors)

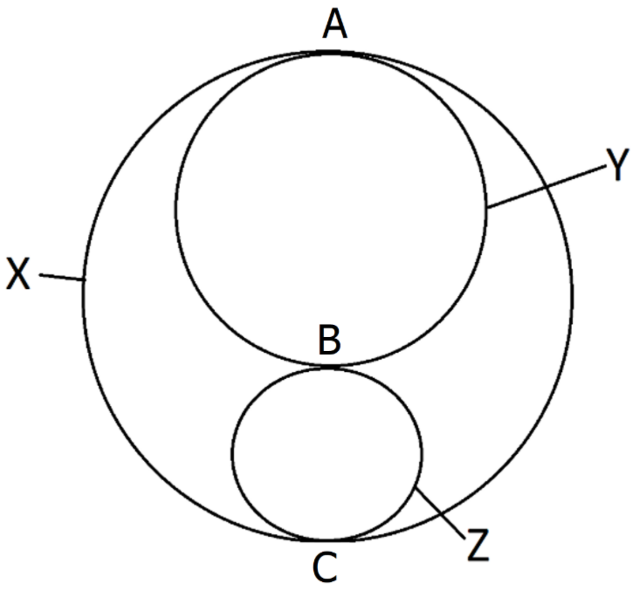
the twelve entries in the final column occur in pairs,

sometimes positive/negative pairs.

So 19291723 is the answer.

Final answer = 1532.

Question 21 (Wed 2nd December)



Roger draws three circles X, Y and Z

that touch at A, B and C as in the diagram.

ABC is a diameter of X, AB is a diameter of Y

and BC is a diameter of Z.

Circle X has area A_x and radius x, circle Y has area A_y and radius y,

and circle Z has area A_z and radius z.

If $A_x + 2A_y + 3A_z = \pi$, what is the largest (to 3s.f.) that x can be?

To get your final answer, multiply your value by 2538, add 1 and take the integer part.

Solution:

Method 1

$$x = y + z \ .$$

$$\pi x^2 + 2\pi y^2 + 3\pi z^2 = \pi \Rightarrow x^2 + 2(x-z)^2 + 3z^2 = 1 \ .$$

Multiplying out, $5z^2 - 4xz + 3x^2 - 1 = 0$.

Solving this as a quadratic in z, $z = \dfrac{4x \pm \sqrt{16x^2 - 20(3x^2 - 1)}}{10} = \dfrac{4x \pm \sqrt{20 - 44x^2}}{10}$.

For z to be real, $20 - 44x^2 \geq 0 \Rightarrow x \leq \sqrt{\dfrac{20}{44}} = \sqrt{\dfrac{5}{11}}$.

Thus the maximum possible value x can take is $\sqrt{\frac{5}{11}} \Rightarrow y = \frac{3}{5}\sqrt{\frac{5}{11}}, z = \frac{2}{5}\sqrt{\frac{5}{11}}$.

So the answer is 0.674 (3s.f.).

Method 2

$y = kx$, $z = (1 - k)x \Rightarrow \pi x^2 + 2\pi k^2x^2 + 3\pi(x - kx)^2 = \pi$.

Thus $x^2(5k^2 - 6k + 4) = 1$.

$5k^2 - 6k + 4$ has a minimum when $k = 0.6$, so x has a maximum of $\sqrt{\frac{5}{11}} = 0.674(3s.f.)$

Final answer = 1711.

Question 22 (Thurs 3rd December)

Mollie sings in a group with five other people.

In a performance, she has to stand next to Tevin.

In addition, she cannot stand next to Becki.

Jonny has to stand on one end of the group of six.

As the audience views things,

in how many different arrangements can the group stand?

To get your final answer, multiply your value by 35.1, add 34 and take the integer part.

Solution:

J stands on one end, leaving five singers.

If M stands next to T,

there are effectively three non-J singers plus the MT pair.

There are 4! ways to arrange these,

multiplying by 2 since M and T can swap for each one.

Multiplying by two for J's two ends gives 96 options.

Now we take away the MB and BM arrangements.

These must appear as BMT or TMB,

which gives 3! + 3!, then times 2 for the two J ends.

So we have 96 – 24 = 72 options.

Final answer = 2561.

Question 23 (Friday 4th December)

Alan is faced with a code, for which he is given a keyword clue.

Flippant and comic, even tongue-in-cheek?

Investigate our useful starters.

523418525-5237 561749 4612135

Alan cracks the code to give a three-digit number; what is this?

To get your final answer, multiply your value by 26.9, add 33 and take the integer part.

Solution:

The first part is an acrostic; the keyword is FACETIOUS.

F	A	C	E	T	I	O	U	S
1	2	3	4	5	6	7	8	9
B	D	G	H	J	K	L	M	N
10	11	12	13	14	15	16	17	18
P	Q	R	V	W	X	Y	Z	
19	20	21	22	23	24	25	26	

Decoding we have TWENTY-TWO TIMES EIGHT = 176.

Final answer = 4767.

Question 24 (Monday 7^{th} December)

Troppo the clown puts on his shoes and socks using ‘a routine’.

A routine involves choosing a foot,

putting a shoe or sock onto that foot,

and then repeating this three further times.

At the end of a routine, he is always wearing

two shoes and two socks. A shoe is always over a sock.

Troppo has four pairs of socks (yellow, orange, purple and white).

He also has three pairs of shoes (blue, red, and green).

His shoes must match, but his socks must not match.

The socks are (unusually) labelled left and right,

and his shoes (as usual) are left and right;

Troppo always obeys these instructions.

How many possible different routines

(where different colours for shoes or socks

count as different routines)

does Troppo have to choose from?

To get your final answer, multiply your value by 329.1, add 150 and take the integer part.

Solution:

Troppo goes either

A. Foot, Sock, Foot, Sock, Foot, Shoe, Foot, Shoe

or

B. Foot, Sock, Foot, Shoe, Foot, Sock, Foot, Shoe.

A.	Foot	Sock	Foot	Sock	Foot	Shoe	Foot	Shoe
Choices	2	4	1	3	2	3	1	1
B.	Foot	Sock	Foot	Shoe	Foot	Sock	Foot	Shoe

Choices	2	4	1	3	1	3	1	1

Multiplying the numbers in row 2 gives 144, while row 4 gives 72.

So the answer is 144 + 72 = 216.

Final answer = 71235.

The Final Question (Tuesday 8th December)

Stage 1

Welcome to the devilish delights

of the Final Question for Ritangle 2020!

To start, put together your final answers

to Questions 1 to 24 into a long string of digits.

Can you decode this string?

Solution to Stage 1

Question	Final Answer
1. Kanesha	831137
2. Evan	37747
3. Yu Yan	431147
4. Wendy	43561
5. Olivia	2531
6. Ramesh	234317
7. Davina	67712
8. Sita	171147
9. Donald	4311
10. Ian	434783
11. Oscar	1143
12. Paula	71116
13. Haris	1711
14. Aisha	92367
15. Nigel	8347
16. Tom	61723
17. Una	4371
18. Susanna	4797
19. Francois	47736
20. Eleanor	1532
21. Roger	1711
22. Mollie	2561
23. Alan	4767
24. Troppo	71235

The answers decrypt via the plain 'prime number' cipher in the table below.

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97	101
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

This gives the message

welldoneoncrackingstageonenowenterthiswordintoyourpageacrostic

i.e.

Well done on cracking Stage One.
Now enter this word into your page: acrostic.

When entered into the appropriate Ritangle page, the word 'acrostic' (which is not guessable until they've solved question 24) leads to the Stage 2 problems.

The first letters of each question from 1 to 24 form an acrostic, which gives the keywords required for the next two ciphers (Diophantus and Fermat).

Stage 2

Solve the eight problems, then decrypt the answers twice to get an eight-letter word. Input this to the appropriate Ritangle page to release the Stage 3 task.

The Eight Questions

Question A

Pete walks from (0, 0) to (x, 100) in a straight line,

and then from (x, 100) to (1000, 0) in a straight line,

where figures are in metres, and where x is 'on the flat' and y is 'uphill'.

You are given that $100 < x < 900$.

Pete's speed is 1 - m metres per second

when he is walking along a line of gradient m.

(So his speed is more than 1 for m negative,

and less than 1 for m positive).

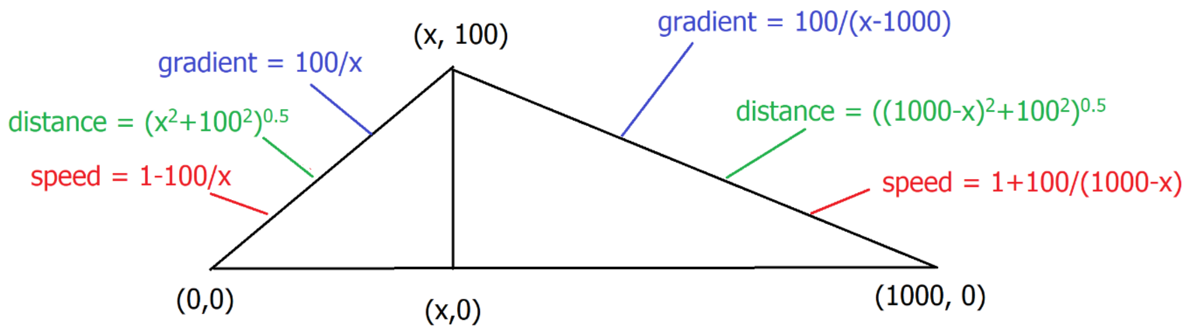
What is the shortest possible time (to 4 s.f.) for Pete's journey in seconds?

A graphing program may be needed here to solve an equation.

To get your final answer,

- ***multiply your value by 40.11, add 1007 and take the integer part.***
- ***multiply your value by 690.11, add 395 and take the integer part.***
- ***multiply your value by 692.11, add 505 and take the integer part.***
- ***concatenate these answers to give a 17-digit integer.***

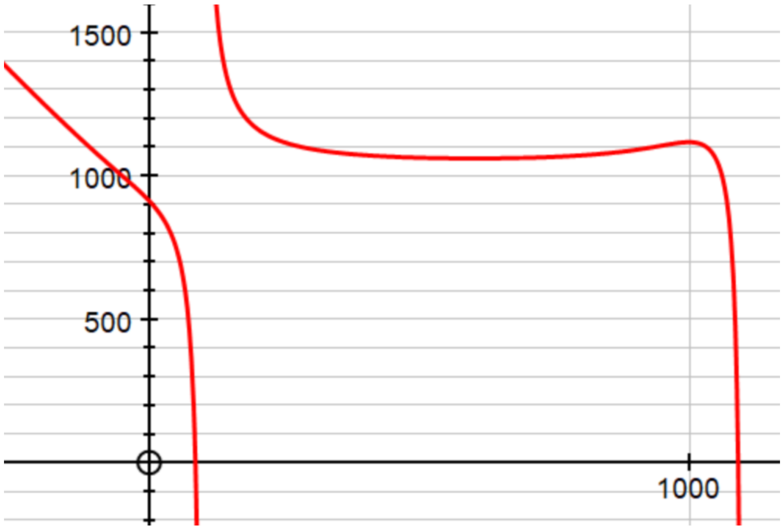
Solution:



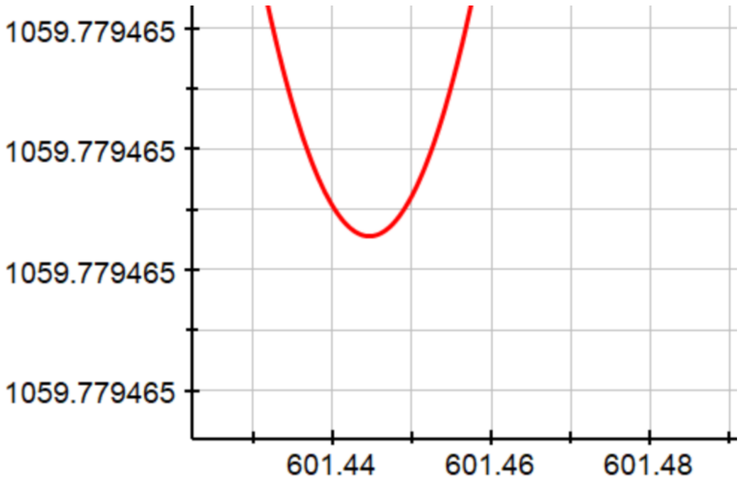
Thus the time taken for the journey is

$$t = \frac{\sqrt{x^2 + 100^2}}{1 - \frac{100}{x}} + \frac{\sqrt{(1000 - x)^2 + 100^2}}{1 + \frac{100}{1000 - x}} \quad .$$

Drawing the graph of t against x gives this;



So there is a minimum for t at around x = 601.44.



This gives the minimum value for t to be 1060 seconds (4s.f.)

Final answer = 43523731911734141

Question B

Olga has two favourite radio stations, Classical Radio, and Radio Classics.

On one particular day, Classical Radio plays a loop of ten pieces of music with no chat in between. The pieces can be called A_1, A_2, ..A_{10}, played in that order, where the piece A_i takes i minutes.

On the same day, Radio Classics plays a loop of 20 pieces of music, A_1 to A_{10} in order followed by A_{11} to A_{20} in order. Once again, the piece A_i takes i minutes each time, and there is no chat between pieces.

At midnight the two loops are set rolling from random starting points (which might not be at the start of a piece).

Olga turns on Classical Radio at 9am to find some piece playing, before turning immediately over to Radio Classics. What is the probability that she finds the same piece playing?

To get your final answer,

- ***multiply your value by 1116988.1, add 1 and take the integer part.***
- ***multiply your value by 425129.1, add 1 and take the integer part.***
- ***multiply your value by 699509.1, add 1 and take the integer part.***
- ***multiply your value by 580109.1, add 1 and take the integer part.***
- ***concatenate these answers to give a 20-digit integer.***

Solution:

The pieces A_1 to A_{10} take 55 minutes in total. The pieces A_1 to A_{20} take 210 minutes in total.

The probability that A_1 is playing on both channels is $\frac{1}{55}\times\frac{1}{210}$.

The probability that A_2 is playing on both channels is $\frac{2}{55}\times\frac{2}{210}$.

…

The probability that A_{10} is playing on both channels is $\frac{10}{55}\times\frac{10}{210}$.

Thus the probability that the music is the same on both channels is

$$\frac{1}{55}\times\frac{1}{210}\left(\sum_{1}^{10} r^2\right)=\frac{1}{55}\times\frac{1}{210}\times\frac{10\times 11\times 21}{6}=\frac{1}{30}.$$

Final answer = 37233141712331719337

Question C

Leroy is working with two sequences, $u_1, u_2, u_3\ldots$ and $v_1, v_2, v_3\ldots$.

The sequence $u_1, u_2, u_3\ldots$ is arithmetic

with first term 1 and common difference d.

The sequence $v_1, v_2, v_3\ldots$ is geometric

with first term 1 and common ratio 3.

Leroy finds that the infinite series $\frac{u_1}{v_1}+\frac{u_2}{v_2}+\frac{u_3}{v_3}+\cdots$ adds to 9.

What is d?

To get your final answer,

- ***multiply your value by 5173.12, add 10 and take the integer part.***
- ***multiply your value by 1196.12, add 16 and take the integer part.***
- ***multiply your value by 41732.12, add 16 and take the integer part.***
- ***multiply your value by 23311.12, add 30 and take the integer part.***
- ***concatenate these answers to give a 22-digit integer.***

Solution:

Method 1

$$\frac{u_1}{v_1}+\frac{u_2}{v_2}+\frac{u_3}{v_3}+\ldots=\frac{1}{1}+\frac{1+d}{3}+\frac{1+2d}{9}+\frac{1+3d}{27}+\ldots$$

$$=\frac{1}{1}+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots+\frac{d}{3}+\frac{2d}{9}+\frac{3d}{27}+\ldots$$

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}.$$

$$\frac{d}{3}+\frac{2d}{9}+\frac{3d}{27}+\ldots=d\left(\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\ldots\right) \qquad *$$

$$\left(\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\ldots\right)=\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\right)+\left(\frac{1}{9}+\frac{1}{27}+\ldots\right)+\left(\frac{1}{27}+\frac{1}{81}\ldots\right)+\ldots$$

$$=\frac{1/3}{1-1/3}+\frac{1/9}{1-1/3}+\frac{1/27}{1-1/3}\ldots=\frac{1/3}{\left(1-1/3\right)^2}=\frac{3}{4}.$$

$$\text{Thus } \frac{u_1}{v_1}+\frac{u_2}{v_2}+\frac{u_3}{v_3}+\ldots=\frac{3}{2}+d\frac{3}{4}=9 \Rightarrow d=10.$$

Method 2

$$f(x)=\frac{1}{3}x+\frac{1}{9}x^2+\frac{1}{27}x^3+\ldots=\frac{x/3}{1-x/3}(\text{if } |x|<3)=\frac{x}{3-x}$$

$$f'(x)=\frac{1}{3}+\frac{2}{9}x+\frac{3}{27}x^2+\frac{4}{81}x^3+\ldots\Rightarrow f'(1)=\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\ldots$$

$$f'(x)=\frac{(3-x)-x(-1)}{(3-x)^2}=\frac{3}{(3-x)^2}\Rightarrow f'(1)=\frac{3}{4}$$

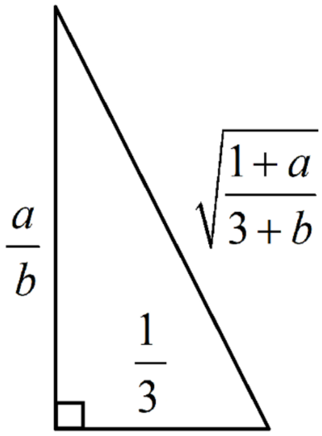
$$\Rightarrow \frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\ldots=\frac{3}{4}.$$

Then as for Method 1. Final answer = 5174111977417337233141

Question D

Yvonne has been challenged to find two positive numbers a and b,

where 2a and 2b are integers, such that a and b satisfy the following diagram.



She notes that a = 4, b = 6 is one possible solution. In this case a + b = 10.

She then finds a second solution, where a = p, b = q and p + q > 10.

What is the smallest possible value for p + q?

You may find an Excel spreadsheet helpful to you here.

To get your final answer,

- ***multiply your value by 655.1, add 66 and take the integer part.***
- ***multiply your value by 4399.1, add 58 and take the integer part.***
- ***multiply your value by 3177.1, add 65 and take the integer part***.
- ***multiply your value by 1468.1, add 74 and take the integer part***.
- ***concatenate these answers to give a 23-digit integer.***

Solution:

Method 1

Certainly $\left(\frac{1}{3}\right)^2+\left(\frac{4}{6}\right)^2=\frac{1+4}{3+6}=\frac{5}{9}$.

In general, $\left(\frac{a}{b}\right)^2+\left(\frac{1}{3}\right)^2=\frac{1+a}{3+b} \Rightarrow 9(3+b)a^2+b^2(b+3)=9b^2(a+1)$.

Rearranging this into a quadratic in a, we get

$9a^2(b + 3) - 9ab^2 + b^2(b - 6)=0$.

Solving using the formula gives $a = \frac{3b^2 \pm b\sqrt{5b^2 + 12b + 72}}{6(b+3)}$.

Now we can put these two expressions for a into a spreadsheet.

b	**a+**	**a-**
0.5	0.247673	-0.17624
1	0.518083	-0.26808
1.5	0.809017	-0.30902
2	1.118022	-0.31802
2.5	1.442682	-0.30632
3	1.780776	-0.28078
3.5	2.130351	-0.24574
4	2.489727	-0.20401
4.5	2.857481	-0.15748
5	3.232419	-0.10742
5.5	3.613538	-0.05471
6	4	0
6.5	4.391101	0.056267
7	4.786248	0.113752
7.5	5.184941	0.172202

48.5	40.32622	5.348539
49	40.76091	5.412163
49.5	41.19564	5.475787
50	41.6304	5.539413
50.5	42.06518	5.603039
51	42.5	5.666667
51.5	42.93484	5.730295
52	43.36971	5.793924
52.5	43.80461	5.857553
53	44.23953	5.921184
53.5	44.67448	5.984815
54	45.10945	6.048447

On the left is the start of the spreadsheet,

with the a = 4, b = 6 solution.

The values a and b both have to be a whole number of halves.

The next time this happens is when a = 42.5, b = 51.

Thus the answer is 93.5.

Method 2

Let A = 2a, B = 2b.

Then $\frac{A^2}{B^2} + \frac{1}{9} = \frac{2+A}{6+B} \Rightarrow 9(6+B)A^2 + B^2(6+B) = 9B^2(2+A)$

This means that $B^2(6+B)$ is divisible by 9, so 3 divides B.

Also, if A and B are both odd, we have a contradiction,

and if B is odd and A is even we also have a contradiction.

Thus B is even and we can say B = 6k.

Substituting in we have $(1+k)A^2+4k^2(1+k)=6k^2(2+A)$

Since hcf(1+k,k) = 1, k^2 divides A^2 and so k divides A.

This we can write A = mk.

Substituting in we have $k(m^2 + 4 - 6m) = 12 - 4 - m^2$

Thus $k = \frac{8 - m^2}{(m-3)^2 - 5}$ where k and m are positive integers.

Now we can try out values of m.

m	1	2	3	4	5	6
k	7/(-1)	4/(-4)	-1/(-5)	-8/(-4)=2	-17/(-1)=17	-28/(4)
Possible?	No	No	No	Yes	Yes	No

k is always negative from here on.

So we have m = 4, k=2, A = 8, B = 12, a = 4, b = 6, or

m = 5, k = 17, A = 85, B = 102, a = 42.5, b = 51.

This method shows that these are in fact

the only solutions to the problem.

Final answer = 61317411373297123137341

Question E

Boris writes down a seven–digit number and calls it n.

He then creates a six-digit number m by deleting one of the digits of n.

He then adds m and n to get the answer 7654321.

What is n?

To get your final answer,

- ***multiply your value by 0.102, add 2549 and take the integer part.***
- ***multiply your value by 0.0392, add 359 and take the integer part.***
- ***multiply your value by 0.0134, add 134 and take the integer part.***
- ***multiply your value by 0.0764, add 92 and take the integer part.***
- ***concatenate these answers to give a 23-digit integer.***

Solution:

Method 1

Let n be abcdefg.

The value m + n is odd, so the deleted digit must be g.

$$\begin{array}{r} a\,b\,c\,d\,e\,f\,g \\ +\;a\,b\,c\,d\,e\,f \\ \hline 7\,6\,5\,4\,3\,2\,1 \end{array}$$

n = 10m + g, so 10m + g + m = 7654321.

7654321 = 11 x 695847 + 4.

Thus n = 6958474.

Check; 6958474 + 695847 = 7654321.

Method 2

We can draw up a table showing possible values.

C denotes a carry.

g	0	1	2	3	4	5	6	7	8	9
f	1	0	9C	8C	7C	6C	5C	4C	3C	2C
e	1	2	2C	3C	4C	5C	6C	7C	8C	9C
d	2	1	0	9C	8C	7C	6C	5C	4C	3C
c	2	3	4	4C	5C	6C	7C	8C	9C	0
b	3	2	1	0	9C	8C	7C	6C	5C	4C
a	3	4	5	6	6C	7C	8C	9C	0	1
a	7	7	7	7	6	6	6	6	6	6

The only column where the last two rows agree is g = 4.

This tells us n = 6958474.

Final answer = 71231327313193377531719

Question F

Idris is considering the equation $a = b^2 - c^2$,

where a is an odd positive integer,

and b and c are positive integers, $b > c$.

She notices that $a = 63$ satisfies this equation in three different ways.

$63 = 32^2 - 31^2$,

$63 = 12^2 - 9^2$,

$63 = 8^2 - 1^2$.

She goes on to find the smallest odd number a

that satisfies this equation in four different ways.

What is this number a?

To get your final answer,

- ***multiply your value by 1296.1, add 81 and take the integer part.***
- ***multiply your value by 108.1, add 11 and take the integer part.***
- ***multiply your value by 302.1, add 21 and take the integer part.***
- ***concatenate these answers to give a 16-digit integer.***

Solution:

Examining the case for 63 suggests that

we seek a number with lots of small odd factors.

We know $b^2 - c^2 = (b + c)(b - c)$.

If $a = pq$ $(p > q)$ where p and q are odd,

then $p = b + c$, $q = b - c \Rightarrow (p + q)/2 = b$, $(p - q)/2 = c$.

We need a number with eight distinct odd factors.

How about $105 = 3 \times 5 \times 7$?

$105 = 1 \times 105 = 53^2 - 52^2$

$= 3 \times 35 = 19^2 - 16^2$

$= 5 \times 21 = 13^2 - 8^2$

$= 7 \times 15 = 11^2 - 4^2$

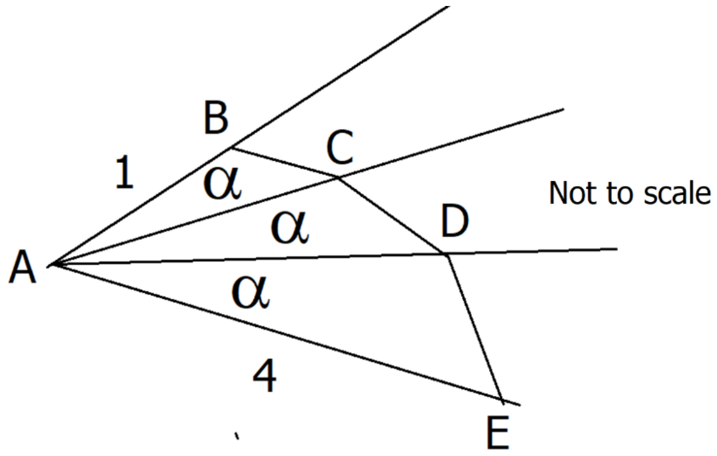
Confirmation that 105 is the answer we seek can come from careful checking of an Excel spreadsheet.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
1	0	3	8	15	24	35	48	63	80	99	120	143	168	195	224	255	288	323	360	399	440	483	528	575	624	675	728	783	840	899	960	1023	1088	1155	1224	1295
2	-3	0	5	12	21	32	45	60	77	96	117	140	165	192	221	252	285	320	357	396	437	480	525	572	621	672	725	780	837	896	957	1020	1085	1152	1221	1292
3	-8	-5	0	7	16	27	40	55	72	91	112	135	160	187	216	247	280	315	352	391	432	475	520	567	616	667	720	775	832	891	952	1015	1080	1147	1216	1287
4	-15	-12	-7	0	9	20	33	48	65	84	105	128	153	180	209	240	273	308	345	384	425	468	513	560	609	660	713	768	825	884	945	1008	1073	1140	1209	1280
5	-24	-21	-16	-9	0	11	24	39	56	75	96	119	144	171	200	231	264	299	336	375	416	459	504	551	600	651	704	759	816	875	936	999	1064	1131	1200	1271
6	-35	-32	-27	-20	-11	0	13	28	45	64	85	108	133	160	189	220	253	288	325	364	405	448	493	540	589	640	693	748	805	864	925	988	1053	1120	1189	1260
7	-48	-45	-40	-33	-24	-13	0	15	32	51	72	95	120	147	176	207	240	275	312	351	392	435	480	527	576	627	680	735	792	851	912	975	1040	1107	1176	1247
8	-63	-60	-55	-48	-39	-28	-15	0	17	36	57	80	105	132	161	192	225	260	297	336	377	420	465	512	561	612	665	720	777	836	897	960	1025	1092	1161	1232
9	-80	-77	-72	-65	-56	-45	-32	-17	0	19	40	63	88	115	144	175	208	243	280	319	360	403	448	495	544	595	648	703	760	819	880	943	1008	1075	1144	1215
10	-99	-96	-91	-84	-75	-64	-51	-36	-19	0	21	44	69	96	125	156	189	224	261	300	341	384	429	476	525	576	629	684	741	800	861	924	989	1056	1125	1196
11	-120	-117	-112	-105	-96	-85	-72	-57	-40	-21	0	23	48	75	104	135	168	203	240	279	320	363	408	455	504	555	608	663	720	779	840	903	968	1035	1104	1175
12	-143	-140	-135	-128	-119	-108	-95	-80	-63	-44	-23	0	25	52	81	112	145	180	217	256	297	340	385	432	481	532	585	640	697	756	817	880	945	1012	1081	1152
13	-168	-165	-160	-153	-144	-133	-120	-105	-88	-69	-48	-25	0	27	56	87	120	155	192	231	272	315	360	407	456	507	560	615	672	731	792	855	920	987	1056	1127
14	-195	-192	-187	-180	-171	-160	-147	-132	-115	-96	-75	-52	-27	0	29	60	93	128	165	204	245	288	333	380	429	480	533	588	645	704	765	828	893	960	1029	1100
15	-224	-221	-216	-209	-200	-189	-176	-161	-144	-125	-104	-81	-56	-29	0	31	64	99	136	175	216	259	304	351	400	451	504	559	616	675	736	799	864	931	1000	1071
16	-255	-252	-247	-240	-231	-220	-207	-192	-175	-156	-135	-112	-87	-60	-31	0	33	68	105	144	185	228	273	320	369	420	473	528	585	644	705	768	833	900	969	1040
17	-288	-285	-280	-273	-264	-253	-240	-225	-208	-189	-168	-145	-120	-93	-64	-33	0	35	72	111	152	195	240	287	336	387	440	495	552	611	672	735	800	867	936	1007
18	-323	-320	-315	-308	-299	-288	-275	-260	-243	-224	-203	-180	-155	-128	-99	-68	-35	0	37	76	117	160	205	252	301	352	405	460	517	576	637	700	765	832	901	972
19	-360	-357	-352	-345	-336	-325	-312	-297	-280	-261	-240	-217	-192	-165	-136	-105	-72	-37	0	39	80	123	168	215	264	315	368	423	480	539	600	663	728	795	864	935
20	-399	-396	-391	-384	-375	-364	-351	-336	-319	-300	-279	-256	-231	-204	-175	-144	-111	-76	-39	0	41	84	129	176	225	276	329	384	441	500	561	624	689	756	825	896
21	-440	-437	-432	-425	-416	-405	-392	-377	-360	-341	-320	-297	-272	-245	-216	-185	-152	-117	-80	-41	0	43	88	135	184	235	288	343	400	459	520	583	648	715	784	855
22	-483	-480	-475	-468	-459	-448	-435	-420	-403	-384	-363	-340	-315	-288	-259	-228	-195	-160	-123	-84	-43	0	45	92	141	192	245	300	357	416	477	540	605	672	741	812
23	-528	-525	-520	-513	-504	-493	-480	-465	-448	-429	-408	-385	-360	-333	-304	-273	-240	-205	-168	-129	-88	-45	0	47	96	147	200	255	312	371	432	495	560	627	696	767
24	-575	-572	-567	-560	-551	-540	-527	-512	-495	-476	-455	-432	-407	-380	-351	-320	-287	-252	-215	-176	-135	-92	-47	0	49	100	153	208	265	324	385	448	513	580	649	720
25	-624	-621	-616	-609	-600	-589	-576	-561	-544	-525	-504	-481	-456	-429	-400	-369	-336	-301	-264	-225	-184	-141	-96	-49	0	51	104	159	216	275	336	399	464	531	600	671
26	-675	-672	-667	-660	-651	-640	-627	-612	-595	-576	-555	-532	-507	-480	-451	-420	-387	-352	-315	-276	-235	-192	-147	-100	-51	0	53	108	165	224	285	348	413	480	549	620
27	-728	-725	-720	-713	-704	-693	-680	-665	-648	-629	-608	-585	-560	-533	-504	-473	-440	-405	-368	-329	-288	-245	-200	-153	-104	-53	0	55	112	171	232	295	360	427	496	567
28	-783	-780	-775	-768	-759	-748	-735	-720	-703	-684	-663	-640	-615	-588	-559	-528	-495	-460	-423	-384	-343	-300	-255	-208	-159	-108	-55	0	57	116	177	240	305	372	441	512
29	-840	-837	-832	-825	-816	-805	-792	-777	-760	-741	-720	-697	-672	-645	-616	-585	-552	-517	-480	-441	-400	-357	-312	-265	-216	-165	-112	-57	0	59	120	183	248	315	384	455
30	-899	-896	-891	-884	-875	-864	-851	-836	-819	-800	-779	-756	-731	-704	-675	-644	-611	-576	-539	-500	-459	-416	-371	-324	-275	-224	-171	-116	-59	0	61	124	189	256	325	396

105

Final answer = 1361711136131741

Question G



Ulysses draws the above diagram,

where the angles BAC, CAD and DAE are all equal to α degrees.

AB = 1, AC = 2, AD = 3, AE = 4.

He draws this so that the lengths BC, CD and DE

are in geometric progression.

What is the angle α to 3s.f.?

To get your final answer,

- ***multiply your value by 2598, add 5 and take the integer part.***
- ***multiply your value by 1743, add 3 and take the integer part.***
- ***multiply your value by 1428, add 23 and take the integer part.***
- ***multiply your value by 1327, add 20 and take the integer part.***
- ***concatenate these answers to give a 20-digit integer.***

Solution:

Let BC = a, CD = ar, De = ar^2. Using the cos rule three times,

$$a^2 = 1 + 4 - 4\cos\alpha,$$

$$a^2 r^2 = 4 + 9 - 12\cos\alpha,$$

$$a^2 r^4 = 9 + 16 - 24\cos\alpha.$$

So on eliminating a and r, $(5-4c)(25-24c) = (13-12c)^2.$

Multiplying out and simplifying gives

$$12c^2 - 23c + 11 = 0 \Rightarrow (c-1)(12c-11) = 0$$

$$\Rightarrow \cos\alpha = \frac{11}{12} \Rightarrow \alpha = 23.6(3s.f.)$$

Final answer = 61317411373372331337

Question H

Sofea chooses four cards at random from a very large number of cards, where hearts, spades, diamonds and clubs are represented equally.

She picks h hearts, s spades, c clubs and d diamonds where h + s + c + d = 4, and calculates V, the variance of the four scores h, s, c, and d.

Here variance is defined as $\dfrac{\sum x^2 - n\left(\bar{x}\right)^2}{n-1}$.

For example, if Sofea picked HSHD,

then the variance would be $\dfrac{(2^2+1^2+1^2+0^2)-4(1)^2}{3}=\dfrac{2}{3}$.

What is the expectation of V?

To get your final answer,

- ***multiply your value by 435236.1, add 1 and take the integer part.***
- ***multiply your value by 3712312.1, add 1 and take the integer part.***
- ***multiply your value by 7319327.1, add 10 and take the integer part.***
- ***concatenate these answers to give a 20-digit integer.***

Solution:

There are five possible distributions for the four cards amongst the suits.

4-0-0-0 will appear four times when considering all possible choices. The variance in this case will be 4.

Similarly 1-1-1-1 will appear 24 times in all possible choices. The variance here is 0.

3-1-0-0, 2-2-0-0, and 2-1-1-0 are the other possible distributions, as in the table below.

					V		V
				Sum x^2	Variance	Cases	Variance*cases
4	0	0	0	16	4	4	16
3	1	0	0	10	2	48	96
2	2	0	0	8	1.333333	36	48
2	1	1	0	6	0.666667	144	96
1	1	1	1	4	0	24	0
						256	256

Average value for V = 256/256 = 1.

The expected value for V is (Σ(V*cases))/(total cases) = 1.

The 256 cases are all equally likely.

Final answer = 43523737123137319337

Stage 2 Solutions

Question	Final Answer	Decrypt 1	Interpret
25. Pete	43523731911734141	FourThree	(4,3)
26. Olga	37233141712331719337	CubeQuintic	(4,5)
27. Leroy	5174111977417337233141	OneHypercube	(1,5)
28. Yvonne	61317411373297123137341	LinearSquare	(1,3)
29. Boris	71231327313193377531719	QuadraticPoint	(2,1)
30. Idris	1361711136131741	AlphaLine	(1,2)
31. Ulysses	61317411373372331337	LinearCubic	(1,3)
32. Sofea	43523737123137319337	FourQuartic	(4,4)

The acrostic on the Stage 2 questions spells POLYBIUS.

The numerical answers decrypt via the ‘prime number’ cipher, this time with keyword DIOPHANTUS (as given in the Stage 1 acrostic) to give the words in the ‘Decrypt 1’ column.

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97	101
d	i	o	p	h	a	n	t	u	s	b	c	e	f	g	j	k	l	m	q	r	v	w	x	y	z

These in turn indicate digits 1 to 5, according to the sets in the table below, and give the pairs in the 'Interpret' column.

1	2	3	4	5
One	Two	Three	Four	Five
Linear	Quadratic	Cubic	Quartic	Quintic
Point	Line	Square	Cube	Hypercube
Alpha	Beta	Gamma	Delta	Epsilon

The ordered pairs identify eight letters (line, column) in the Polybius square with keyword FERMAT, which is also given in the Stage 1 acrostic:

	1	2	3	4	5
1	F	E	R	M	A
2	T	B	C	D	G
3	H	I/J	K	L	N
4	O	P	Q	S	U
5	V	W	X	Y	Z

The Polybius decryption leads to the eight-letter word **quarters**. When inputted to the appropriate Ritangle sheet, **quarters** leads to Stage 3.

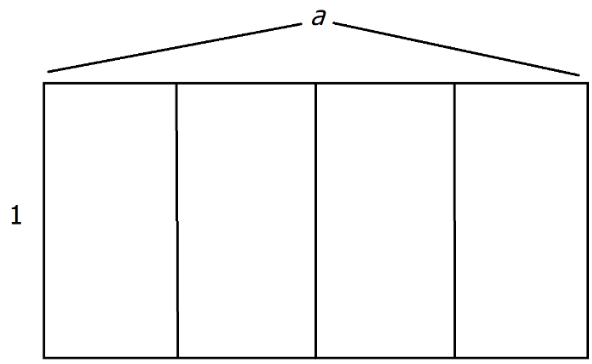
Stage 3

This is the final task for Ritangle 2020.

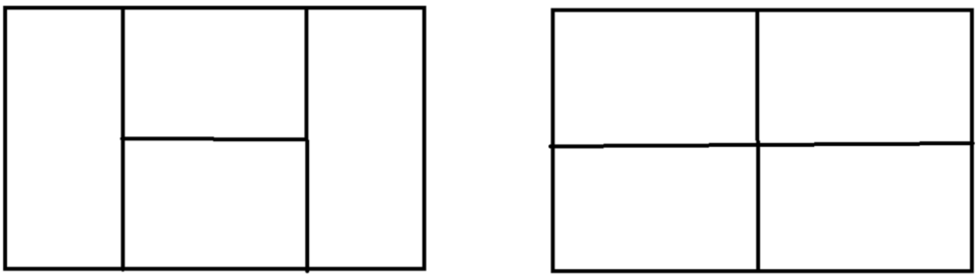
Rectangle Division

You have a rectangle with a vertical side of length 1 and a horizontal side of length *a* (≥ 1). You need to dissect it into four pieces, each with equal area = $\frac{a}{4}$, using lines with total length *L*. What is the smallest *L* can be?

For example, three vertical lines will accomplish the division, with total length *L* = 3, as shown in the upper figure. If *a* is large enough then this will always be the minimum-length solution. We can call this the Trident Solution, where *L* = 3.



But other dissections are possible. On the left below we have the H Solution, where $L = \frac{a}{2}+2$, while on the right we have the Cross Solution, where *L* = *a* + 1.



Is it possible to improve on these solutions? The internal lines don't have to be horizontal or vertical or even straight. You need to find the least possible internal line length *L*

1. for a = 1 (the square), and
2. for a = 2.

Give your answers to four decimal places. It may be that you will find a better solution than the one we have, in which case we will need to see your working!

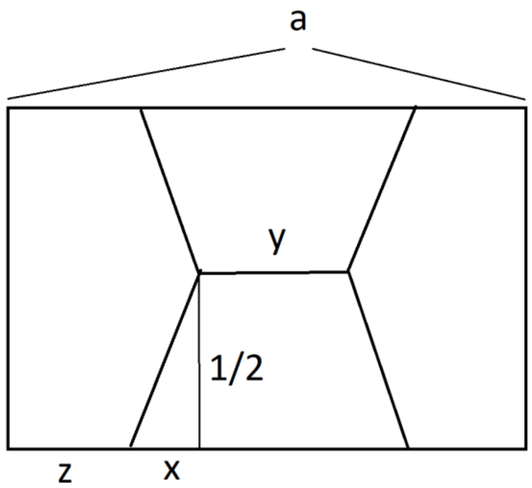
Solution

Throughout this question, it helps to know about the mathematics of soap films and how they can be used to solve this type of mathematical problem. Try searching on Youtube for 'the mathematics of soap films'.

For *a* = 1;

The Trident Solution gives us *L* = 3, while the H Solution gives us *L* = 2.5, and the Cross Solution gives us *L* = 2. Can we improve on this?

We know from looking at the shapes soap films take up that angles of $\frac{2\pi}{3}$ are likely to prove profitable. Let's suppose the following diagram with $\frac{2\pi}{3}$ angles gives us what we shall call the Slanted H Solution for general *a*.



$$2x + 2z + y = a \Rightarrow y = a - 2x - 2z$$

$$\frac{1}{2}(x + y) = \frac{a}{4} \Rightarrow 2x + 2y = a$$

$$\Rightarrow 2x + 2a - 4x - 4z = a \Rightarrow z = \frac{a}{4} - \frac{x}{2} \Rightarrow y = \frac{a}{2} - x.$$

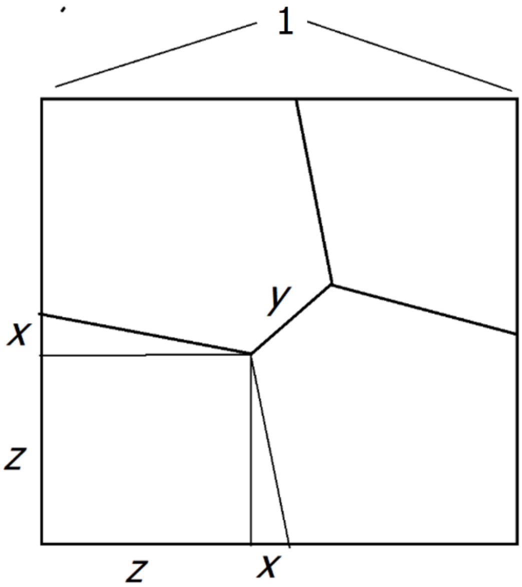
$$L = y + 4\sqrt{x^2 + \frac{1}{4}} = \frac{a}{2} - x + 4\sqrt{x^2 + \frac{1}{4}}$$

$$\frac{dL}{dx} = -1 + 4\frac{x}{\sqrt{x^2 + \frac{1}{4}}} = 0 \text{ when } 60x^2 = 1$$

$$\Rightarrow \mathrm{L}_{\min} = \frac{a}{2} + \frac{\sqrt{15}}{2}.$$

So if *a* = 1 (checking that x, y and z are all positive), we get 2.4365, which is not as good as the Cross Solution.

But this gives us an idea. How about this symmetrical solution for *a* = 1 with $\frac{2\pi}{3}$ angles? We'll say this arrangement gives the Rotated Slanted H Solution (angles = $\frac{2\pi}{3}$).



We have these three equations;

$$y + 2z\sqrt{2} = \sqrt{2}\ , \qquad z^2 + zx = 0.25\ , \qquad \tan\frac{\pi}{12} = \frac{x}{z}\ .$$

which give us $L = 4\sqrt{x^2 + z^2} + y = 1.9971$.

This is the best solution we have so far.

Can we improve this if we don't insist on the $\frac{2\pi}{3}$ angles?

Let's call this the Rotated Slanted H Solution (any angle).

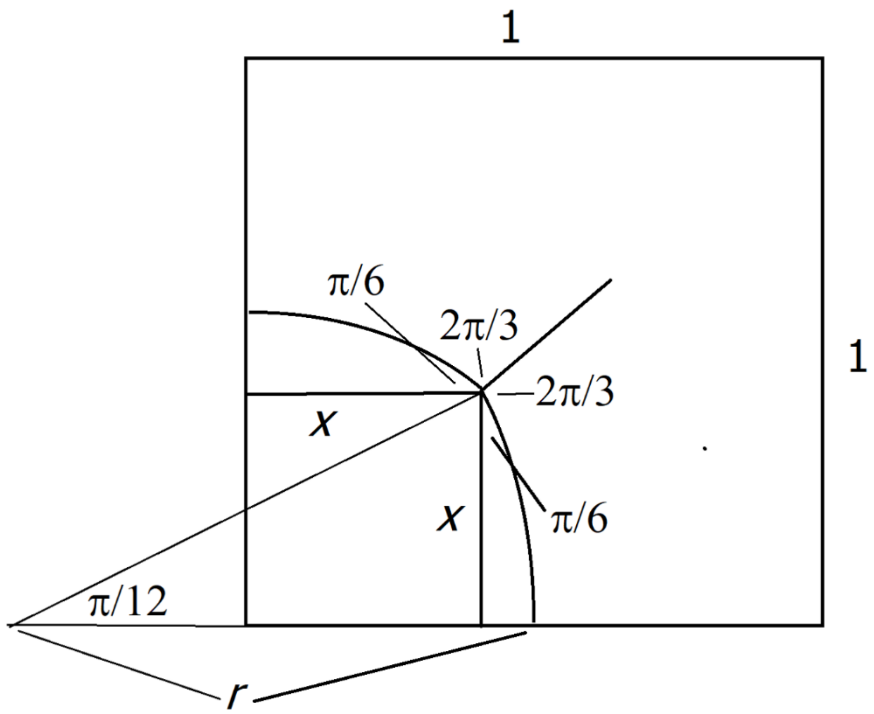
This means we lose the $\tan\frac{\pi}{12} = \frac{x}{z}$ equation, so

$$1 = \frac{\sqrt{(32\cdot z^4 - 8\cdot z^2 + 1)}}{z} - 2\cdot\sqrt{2}\cdot z + \sqrt{2}$$

Differentiating w.r.t. z and putting equal to 0 gives L = 1.9811 for a = 1,

when z = 0.46947, once again an improvement.

However, our soap film knowledge suggests that a symmetrical solution with circular arcs is going to be better still. We can call this Rotated Slanted H with Bubbles Solution.



We have here two circular arcs, each with radius r, giving the two equations

$$x = r\sin\frac{\pi}{12}, x^2 + 2\left(\frac{\pi r^2}{24} - \frac{1}{2}xr\cos\frac{\pi}{12}\right) = \frac{1}{4} \ .$$

Now $L = 4r\frac{\pi}{12} + (1-2x)\sqrt{2}$ which gives us L = 1.9756. This is indeed an improvement! This result is the best we managed for L with a = 1.

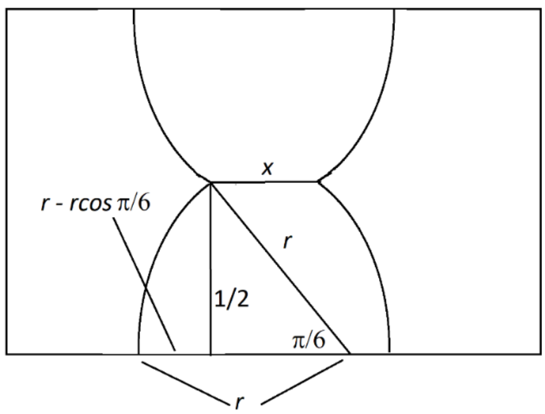
For a = 2;

The Slanted H Solution with a = 2 beats the Trident, H and Cross Solutions

for a = 2.

$$\mathrm{L} = \frac{a}{2} + \frac{\sqrt{15}}{2} = 2.9365 \text{ when } a = 2.$$

However, a soap film tweak on that solution can reduce the total line length further. A soap film will meet a planar surface at right angles, and three soap film lines meet at angles of $\frac{2\pi}{3}$. For our problem, you can imagine each area containing a gas, and then adjusting the pressure in each area to make the areas equal. Doing this keeps the arcs circular. Let's call this the Slanted H with Bubbles Solution.



So the diagram shows four symmetrically placed circular arcs that meet the surrounding rectangle at right angles, and which meet the central line segment of length x at an angle of $\frac{2\pi}{3}$. The four areas are equal, so

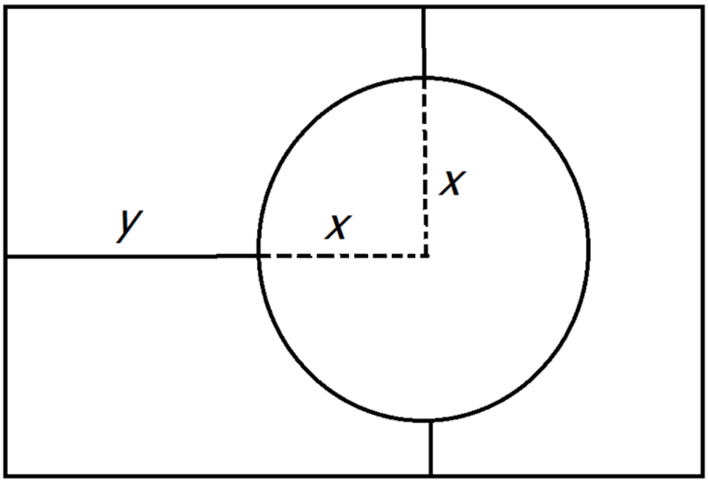
$$\frac{a}{4} = \frac{x}{2} + 2\left(\frac{1}{2}r^2\frac{\pi}{6} - \frac{1}{2}\times\frac{1}{2}r\sin\frac{\pi}{6}\right) \Rightarrow x = \frac{a + \sqrt{3} - \frac{2\pi}{3}}{2} .$$

Also $\sin\frac{\pi}{6} = \frac{1}{2}r \Rightarrow r = 1.$

Now $L = 4\left(r\frac{\pi}{6}\right) + x = \frac{2\pi}{3} + \frac{a}{2} + \frac{\sqrt{3}}{2} - \frac{\pi}{3} = \frac{a}{2} + \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$

For a = 2, this gives *L* = 2.9132, which is a small improvement on 2.9365.

What happens if we try this arrangement (the Circle Solution)?



$L = y + 2\left(\frac{1}{2} - x\right) + 2\pi x$ and $\frac{a}{4} = \pi x^2 = \frac{1}{2}(y + x) - \frac{\pi x^2}{4}$.

$$l = \frac{5 \cdot a}{8} + \frac{\sqrt{a} \cdot (2 \cdot \pi - 3)}{2 \cdot \sqrt{\pi}} + 1$$

$$l = 0.625 \cdot a + 0.9261694755 \cdot \sqrt{a} + 1$$

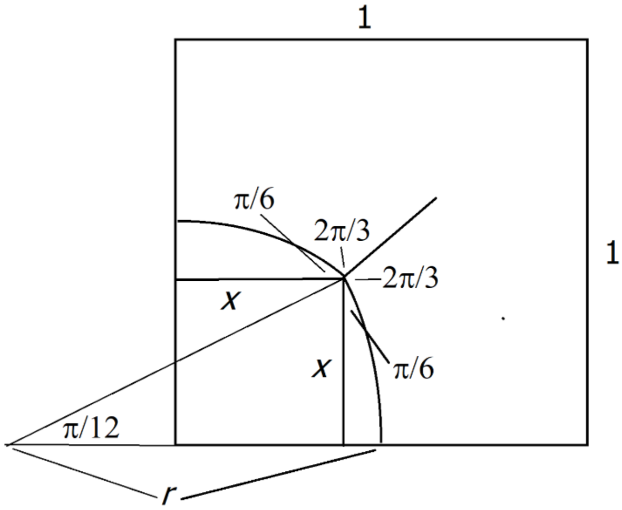
$$a = 2$$

$$l = 3.559801433$$

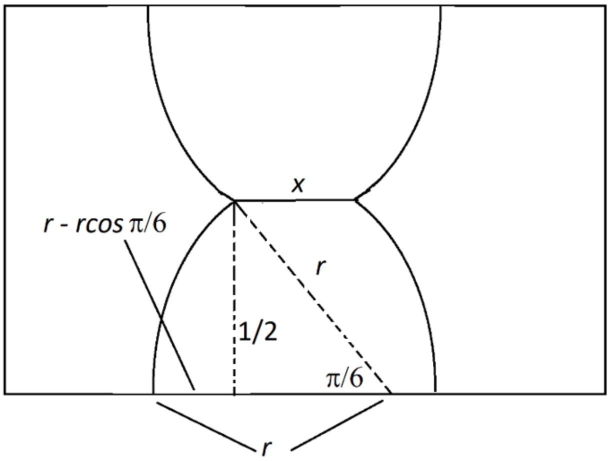
This is a long way off our best answer so far; it turns out that even introducing angles of $\frac{2\pi}{3}$ for the circular arcs here doesn't help things along enough to get a rival result. Try as we might (and there are other arrangements that might suggest themselves), it appears that the Slanted H with Bubbles Solution is the best we can do when a = 2.

So to sum up, here are our best solutions;

a = 1, *L* = 1.9756.



a = 2, *L* = 2.9132.



But it may be that you can be ingenious and find something shorter!