Edexcel international Pure mathematics 2
Trigonometry

Section 1: Trigonometric functions and identities

Notes and Examples

In this section you learn about the trigonometric functions and some trigonometric identities.

These notes contain sub-sections on:

You will already be familiar with using the main trigonometric functions to find missing angles in geometrical contexts, and of the graphs, and their transformations, for sine, cosine and tangent functions.

Trigonometric identities

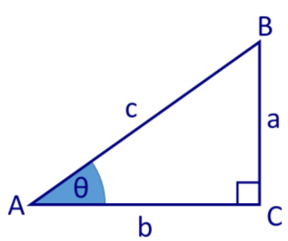
Identities can be used to solve more complex trigonometric equations.

At P2 level you need to know the following identities:

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \sin^2\theta + \cos^2\theta \equiv 1$$

e.g. $\tan 30° = \dfrac{\sin 30°}{\cos 30°}$

An identity is true for all values of θ. An identity is represented by three horizontal lines, ≡, between the two expressions that are identical.



Consider the definitions of each trigonometric ratio in a right angled triangle.

$\cos\theta = \frac{b}{c}$ and $\sin\theta = \frac{a}{c}$ which means $\dfrac{\sin\theta}{\cos\theta} = \dfrac{a/c}{b/c} = \dfrac{a}{b} = \tan\theta$

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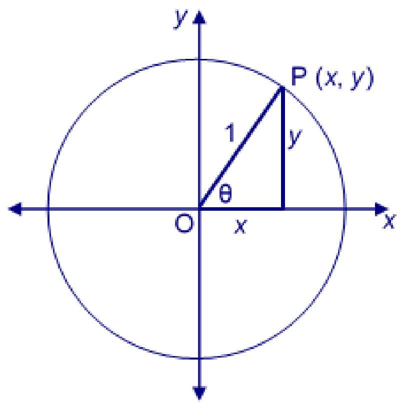
The definitions for sine, cosine and tangent can be extended to angles of any size using a diagram like the one below.

This gives the definitions:

$$\sin\theta = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x}$$



It is possible to see that immediately that $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$,

and Pythagoras' theorem leads to $\sin^2\theta + \cos^2\theta \equiv 1$.

In the next example you need to use the trigonometric identities to rewrite an expression.

Example 1

Show that $(\sin\theta + \cos\theta)(\sin\theta - \cos\theta) = 2\sin^2\theta - 1$

Solution

Working with the LHS and expanding the brackets gives:

$$(\sin\theta + \cos\theta)(\sin\theta - \cos\theta) = \sin^2\theta - \cos^2\theta \qquad ①$$

Since $\sin^2\theta + \cos^2\theta \equiv 1$ then $\cos^2\theta \equiv 1 - \sin^2\theta$ ②

Substituting ② into ① gives:

$$(\sin\theta + \cos\theta)(\sin\theta - \cos\theta) = \sin^2\theta - (1 - \sin^2\theta)$$

Simplifying: $(\sin\theta + \cos\theta)(\sin\theta - \cos\theta) = 2\sin^2\theta - 1$ as required.

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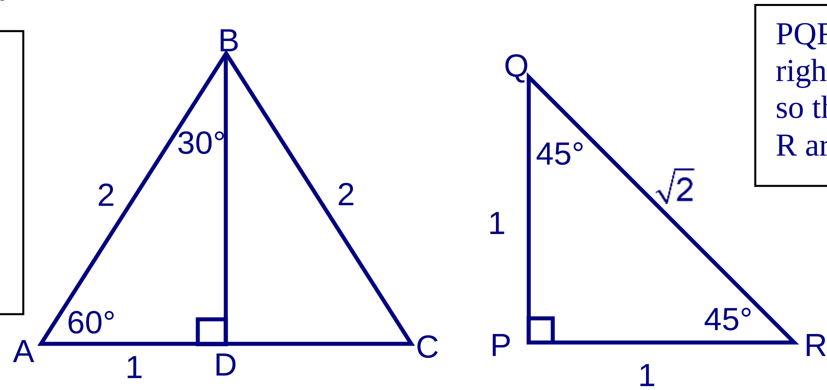
Common values of sin *θ*, cos *θ* and tan *θ*

The two triangles below help you to find the values of sin θ, cos θ and tan θ for θ = 30°, 45°, 60°.

ABC is an equilateral triangle, so all its angles are 60°. D is the midpoint of AC, so that triangle ABD is a right-angled triangle with AD = 1.



PQR is an isosceles right-angled triangles, so the angles at Q and R are 45°.

You should learn the values of sin θ, cos θ and tan θ for θ = 0°, 30°, 45°, 60° and 90°. These are shown in the table below.

θ	0°	30°	45°	60°	90°
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
tan θ	0	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

In the example below you need to substitute exact values into an expression.

Example 2

Show that $\sin^2 30° + \cos^2 30° = 1$

Solution

$\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$

So substituting these values into $\sin^2 30° + \cos^2 30°$:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

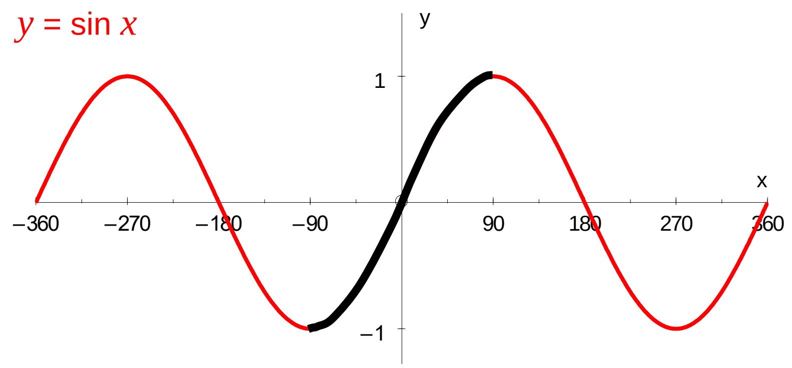
as required.

Principal values

There are infinitely many roots to an equation like $\sin\theta = \frac{1}{2}$.

Your calculator will only give one root – the ***principal value***.
You find this by pressing the calculator keys for arcsin 0.5 (or $\sin^{-1}$ 0.5 or invsin 0.5).
Check that you can get the answer of 30°.

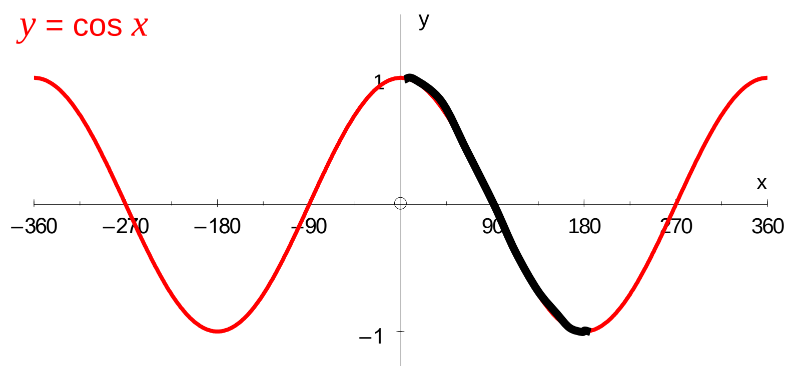
You can find other roots by looking at the symmetry of the appropriate graph.



The principal value for the inverse sine function given by your calculator will be:

$$-90^\circ \le (\sin^{-1} y) \le 90^\circ$$

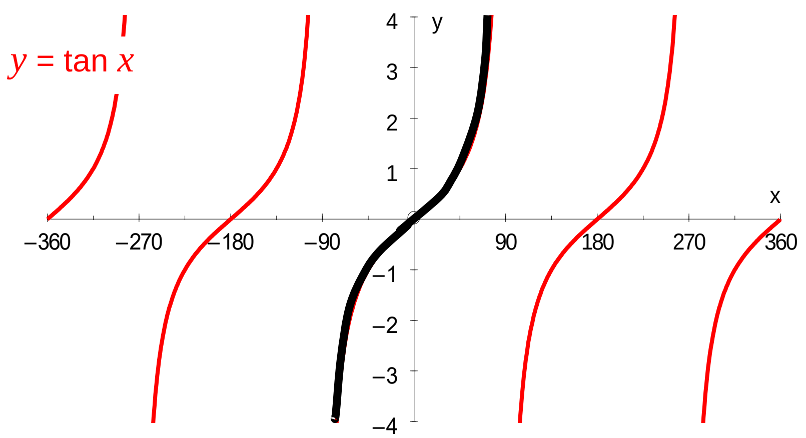
A second root in a 360° can be found using $180^\circ - x$.



The principal value for the inverse cosine function given by your calculator will be:

$$0^\circ \le (\sin^{-1} y) \le 180^\circ$$

A second root in a 360° can be found using $360^\circ - x$



The principal value for the inverse tangent function given by your calculator will be:

$$-90^\circ \le (\sin^{-1} y) \le 90^\circ$$

A second root in a 360° can be found using $x + 180^\circ$ or $x - 180^\circ$

Alternatively, you can use the quadrant diagram (CAST) to find other roots, by thinking about which quadrants the roots will be in.

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Solving simple trigonometric equations

Because there are infinitely many roots to a trigonometric equation you are only ever asked to find a few of them! Any question at this level asking you to solve a trigonometric equation will also give you the interval or range of values in which the roots must lie, e.g. you might be asked to solve $\tan\theta = 2$ for $0° \le \theta \le 360°$.
You can only directly solve trigonometric equations like $\sin\theta = \frac{1}{2}$ or $\cos\theta = \frac{1}{4}$ or $\tan\theta = -2$. Here is an example.

Example 3

Solve $\sin\theta = \dfrac{\sqrt{3}}{2}$ for $-360° \le \theta \le 360°$.

Solution

$\sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = 60°$

There will be a second root in the second quadrant.
180° - 60° = 120° is also a root.
Since $y = \sin\theta$ has a period of 360° any other roots can be found by adding/subtracting 360° to these two roots.
So the other roots are:

60° - 360° = -300°
and 120° - 360° = -240°

So the values of θ for which $\sin\theta = \dfrac{\sqrt{3}}{2}$ are -300°, -240°, 60°, 120°.

Almost all equations of the type in Example 1 have two roots in the range $0° \le x < 360°$.

However, this is not true of all trigonometric equations. Suppose, for example, that you want to solve the equation $\sin 2x = 0.5$ in the range $0° \le x < 360°$. You can find that $2x = 30°$ or $150°$, which means that $x = 15°$ or $75°$. However, there are two further roots in the range $0° \le x < 360°$, given by $2x = 30° + 360° = 390° \Rightarrow x = 195°$, and $2x = 150° + 360° = 510° \Rightarrow x = 255°$. So there are four roots: $x = 15°, 75°, 195°, 255°$.

This means that if you are solving an equation of the form $\sin nx = k$, you need to adjust the range in which you look for initial roots.

Example 4

Solve each of the following equations in the given range:
(i) $\tan 3x = 1$ for $0 \le x < 360°$
(ii) $\cos(2x + 40°) = 0.5$ for $-180° < x \le 180°$

Solution

(i) $\tan 3x = 1$

$3x = 45°,\ 225°,\ 405°,\ 585°,\ 765°,\ 945°$

$x = 15°,\ 75°,\ 135°,\ 195°,\ 255°,\ 315°$

You need to look for roots for $3x$ in the range 0° to 1080°

(ii) $\cos(2x + 40°) = 0.5$

$2x + 40° = -300°,\ -60°,\ 60°,\ 300°$

$2x = -340°,\ -100°,\ 20°,\ 260°$

$x = -170°,\ -50°,\ 10°,\ 130°$

The lower end of the range you need to look in is $-180° \times 2 + 40° = -320°$, and the upper end is $180° \times 2 + 40° = 400°$

An alternative method is to consider the symmetries of the transformed graph within the given range:

Different people find they prefer different methods. It is good to be able to think about things in different ways, so do give both a chance. Ultimately choose the method that makes most sense to you!

Example 5

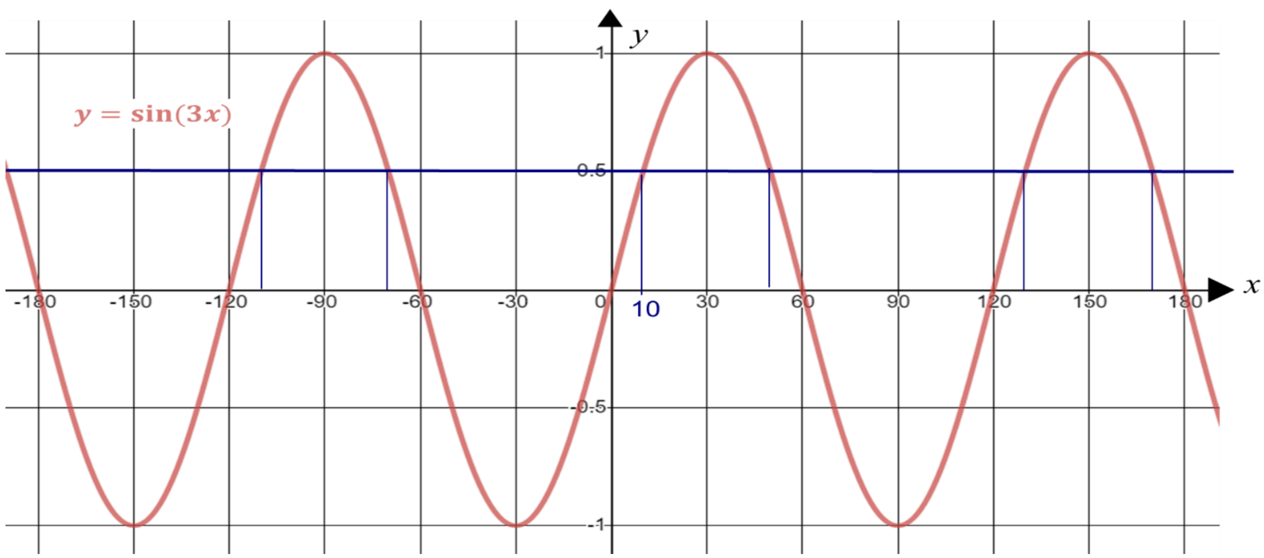
Solve the equation $\sin(3x) = \frac{1}{2},\ \ -180^\circ \le x \le 180^\circ$

Solution

$3x = \sin^{-1}(\tfrac{1}{2}) = 30^\circ$

$x = 10^\circ$

Consider the graph of $y = \sin(3x)$ in the required range:



It is possible to use the symmetries of the transformed graph with one solution to find all the others.

x = -110°, -70°, 10°, 50°, 130°, 170°

More complicated trigonometric equations

Any more complicated equations need to be manipulated algebraically before they can be solved. There are a number of techniques you can use:

1. Rearrange the equation to make $\cos\theta, \sin\theta$ or $\tan\theta$ the subject.
2. Check to see if the equation factorises to give two (or more) equations which involve just one trigonometric function (see Example 6).
   If it is a quadratic in either $\sin\theta$, $\cos\theta$, or $\tan\theta$ it can either be factorised or solved using the formula for solving quadratic equations (see Example 7).
3. If the equation involves just $\sin\theta$ and $\cos\theta$ (and no powers), check to see if you can use the identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ (see Example 8).
4. If the equation contains a mixture of trigonometric functions (e.g. $\cos^2\theta$ and $\sin\theta$) then you may need to use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to make it a quadratic in either $\sin\theta$, $\cos\theta$, or $\tan\theta$ (see Example 9).

Example 6
Solve $2\cos\theta\sin\theta + \cos\theta = 0$ for $0° \le \theta \le 360°$.

Solution
$2\cos\theta\sin\theta + \cos\theta = 0$ can be factorised as there is $\cos\theta$ in both terms on the LHS.
Factorise: $\cos\theta(2\sin\theta + 1) = 0$
So either $\cos\theta = 0$ or $2\sin\theta + 1 = 0$

It is wrong to divide through by $\cos\theta$ because you lose the roots to $\cos\theta = 0$

$\cos\theta = 0 \Rightarrow \theta = 90°$
360° - 90° = 270° is also a root.

$2\sin\theta + 1 = 0 \Rightarrow \sin\theta = -\frac{1}{2}$
This has roots in the third and fourth quadrants.
The roots are 180° + 30° = 210° and 360° - 30° = 330°.

So the values of θ for which $2\cos\theta\sin\theta + \cos\theta = 0$ are 90°, 210°, 270° and 330°.

In example 7 you need to spot the 'hidden quadratic' and solve accordingly.

Example 7
Solve $2\cos^2\theta + 3\cos\theta = 2$ for $0° \le \theta \le 360°$.

Solution
$2\cos^2\theta + 3\cos\theta = 2$ is a quadratic equation in $\cos\theta$
Rearrange the quadratic: $2\cos^2\theta + 3\cos\theta - 2 = 0$

You can replace $\cos\theta$ with x to make things simpler! Or factorise straightaway to get: $(2\cos\theta - 1)(\cos\theta + 2) = 0$ and then solve.

Let $\cos\theta = x$: $2x^2 + 3x - 2 = 0$

Factorise: $(2x-1)(x+2) = 0$

$x = \frac{1}{2}$ or $x = -2 \Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -2$

$\cos\theta = -2$ has no real roots.

So we need to solve $\cos\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = 60°$

There is also a root in the 4th quadrant, so 360° - 60° = 300° is also a root.

So the values of θ for which $2\cos^2\theta + 3\cos\theta = 2$ are 60° and 300°.

In the next example you need to use the identity $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$.

Example 8

Solve $\sin\theta - 2\cos\theta = 0$ for $0° \le \theta \le 360°$.

Solution

You need to rearrange the equation.

$\sin\theta - 2\cos\theta = 0$

Dividing by cos θ: $\dfrac{\sin\theta}{\cos\theta} - 2 = 0$

> You can safely divide by cos θ because it can't be equal to 0. If it were then sin θ would also have to be 0 and cos θ and sin θ are never both 0 for the same value of θ.

Since $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$: $\tan\theta - 2 = 0$

$\Rightarrow \tan\theta = 2$

$\Rightarrow \theta = 63.4°$ to 1 d.p.

There is also a root in the 3rd quadrant.

So 63.4° + 180° = 243.4° is also a root.

So the values of θ for which $\sin\theta - 2\cos\theta = 0$ are 63.4° and 243.4° to 1 d.p.

In the next example you need to use the trigonometric identity $\sin^2\theta + \cos^2\theta \equiv 1$.

Example 9

Solve $\sin^2 x + \sin x = \cos^2 x$ for $0° \le x \le 360°$

Solution

Rearranging the identity $\sin^2\theta + \cos^2\theta \equiv 1$

gives: $\cos^2 x \equiv 1 - \sin^2 x$ ①

Substituting ① into the equation $\sin^2 x + \sin x = \cos^2 x$ gives:

$$\sin^2 x + \sin x = 1 - \sin^2 x$$

This is a quadratic in $\sin x$.

Rearranging: $2\sin^2 x + \sin x - 1 = 0$

Rearranging: $2\sin^2 x + \sin x - 1 = 0$

This factorises to give: $(2\sin x - 1)(\sin x + 1) = 0$

So either: $2\sin x - 1 = 0$ or $\sin x + 1 = 0$

$\Rightarrow \sin x = \frac{1}{2}$ $\Rightarrow \sin x = -1$

$\Rightarrow x = 30°$ or $150°$ $\Rightarrow x = 270°$

So the roots to $\sin^2 x + \sin x = \cos^2 x$ are $x = 30°, 150°$ or $270°$