Ritangle 2021

Questions in black, solutions in red

Questions 1-5 are the preliminary questions, one a week.

Question 1 (Monday October 4th)

The equations below feature six three-digit numbers.

$$abc + def = ghi$$

$$cba + fed = ihg$$

You are given that the digits a to i are

0, 1, 2, 3, 4, 5, 6 and 7 plus an extra 6.

No number starts with a 0.

What is $g \times h \times i$?

To get your final answer, multiply this by 447.2 and take the integer part.

(75129)

Correct answer releases

O

Question 1 Answer

Looking at the end digits in the first equation,

either $c + f = i$, or $c + f = 10 + i$.

Looking at the start digits in the second equation,

$c + f = 10 + i$ is impossible, so $c + f = i$.

Similarly $a + d = g$, which means $b + e = h$.

So we have $a + d = g$, $b + e = h$, $c + f = i$.

0 must be on the left hand side of one of these equations, so

$0 + 6 = 6$ is the only possibility.

7 must be on the right of one equation.

The numbers add to 34, so $g + h + i = 17$,

and so the third number on the right is 4.

Thus we have (in some order)

$0 + 6 = 6$, $2 + 5 = 7$, $1 + 3 = 4$, and $g \times h \times i = 168$.

The equations could be

$201 + 563 = 764$, $102 + 365 = 467$.

Question 2 (Monday October 11th)

Brian is thinking about

- his son Alan's age, A,
- his own age, B, and
- his grandfather Colin's age, C.

He realises that in some order, A, B and C are

- a triangle number,
- a square number and
- a Fibonacci number.

He then realises that in exactly nine years' time, assuming everyone is still around, the same will be true again.

How old is Alan?

Take the Age of Consent to be 16; this is respected in this family at all times!

Assume that no one is older than 100 here.

To get your final answer, multiply this by 13254.1 and take the integer part.

(13254)

Correct answer releases

W

Question 2 Answer

Triangle 0, 1, 3, 6, 10,15,21,28,36,45,55,66,78,91...

+9 9, 10,12,15,19,24,30,37,45,54,64,75,87,100

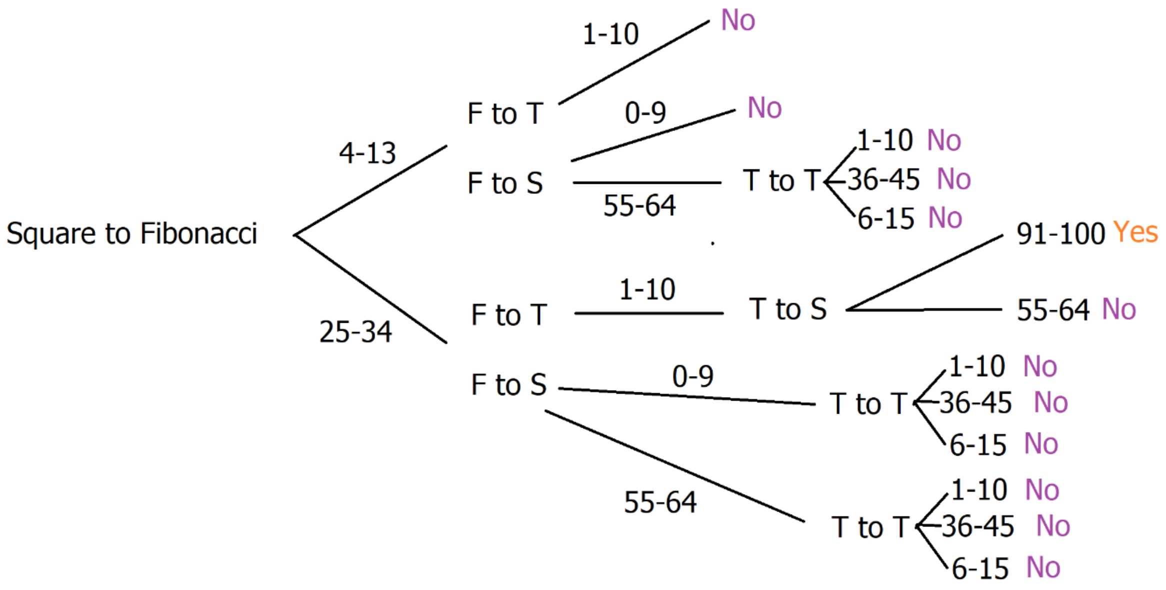
Square 0, 1, 4, 9,16,25,36,49,64,81,100...

+9 9, 10,13,18,25,34,45,58,73,90,109...

Fibonacci 0, 1, 1, 2, 3, 5, 8, 13,21,34,55,89...

+9 9, 10,10,11,12,14,17,22,30,43,64,98...

From the above, the square age must lead on to the Fibonacci age.



So Alan is 1 while Brian muses

(and Colin is 91 and Brian is 25).

We are aware that more than one team didn't spot the significance of the word 'grandfather' and so thought there was more than one possible answer. Reading carefully is an important skill in life and in mathematics!

Question 3 (Monday October 18th)

How do we decide if a number is divisible by 11?

Write down the digit sum, but alternating + and – signs.

The result is divisible by 11 if and only if the original number is.

So given 12345, 1 – 2 + 3 – 4 + 5 = 3,

and since 11 does not go into 3, 11 does not go into 12345.

On the other hand, given 92345, 9 – 2 + 3 – 4 + 5 = 11,

since 11 does go into 11, 11 goes into 92345.

What's the first multiple of 11 to contain all 10 digits?

Note; the number cannot begin with zero.

To get your final answer, multiply this by 0.000025 and take the integer part.

(25609)

Correct answer releases

Z

Question 3 Answer

The number we seek will be at least a ten-digit number.

So we want to split the digits from 0 to 9 into two sets of five digits each

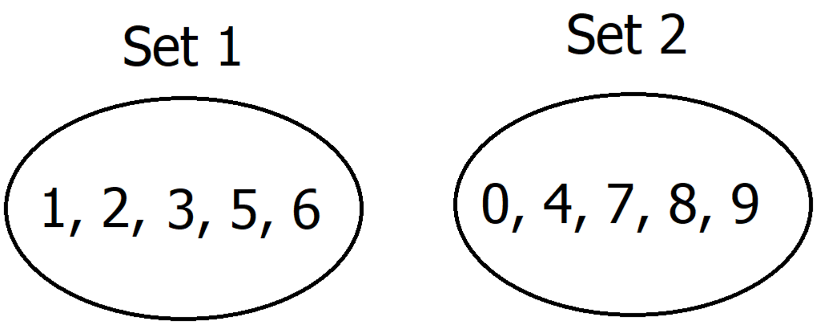
whose sum-difference is 0, or 11, or 22, or…

0 is impossible since 45 is odd.

A difference of 11 means the digits in the two sets

must sum to 17 and 28.

We want the smaller digits at the start. The best we can do is



So our number is 1024375869.

Question 4 (Monday November 1st)

The positive integers a, b and c

are the smallest that satisfy the equations

$$a^5 = 3b^4 = 2c^3.$$

If $\log_{10}a + \log_{10}b + \log_{10}c = p\log_{10}2 + q\log_{10}3$,

where p and q are positive integers, then what is p + q?

To get your final answer, multiply this by 1385.2 and take the integer part.

(91423)

Correct answer releases

q

Question 4 Solution

Let a = 2^r3^s, b = 2^t3^u, c = 2^v3^w.

Then $2^{5r}3^{5s} = 2^{4t}3^{4u+1} = 2^{3v+1}3^{3w}$.

Thus 5r = 4t = 3v + 1.

Smallest possible values are r = 8, t = 10, v = 13.

Also 5s = 4u + 1 = 3w.

Smallest possible values are s = 9, u = 11, w = 15.

Thus a = $2^8 3^9$, b = $2^{10}3^{11}$, c = $2^{13}3^{15}$.

$\log_{10}a + \log_{10}b + \log_{10}c$

$= (8\log_{10}2 + 9\log_{10}3) + (10\log_{10}2 + 11\log_{10}3) + (13\log_{10}2 + 15\log_{10}3)$

$= 31\log_{10}2 + 35\log_{10}3.$

So p = 31, q = 35, p + q = 66.

Question 5 (Monday November 8th)

A is the point (0, 1) while B is the point (2, 0).

C is a point (c, 2c) where c > 0,

and D is the point (d, -d/2) where d > 0.

The triangles ABC and ABD are congruent.

How long is CD?

To get your final answer,

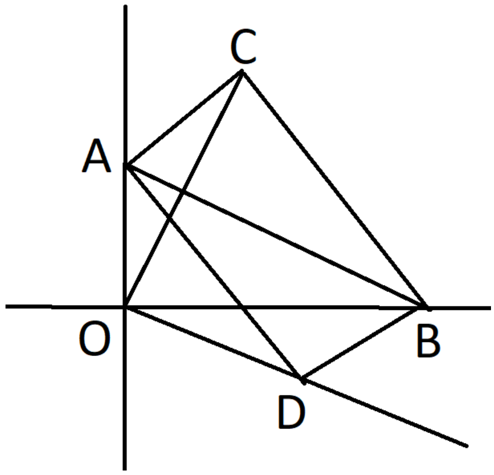
multiply this answer by 8880.3 and take the integer part.

(19856)

Correct answer releases

i

Question 5 Answer



C is on y = 2x, while D is on y = -x/2.

Since OD is parallel to AB, as D varies the area of ABD is constant.

This value must be 1 (imagine D tending to O.)

So the area of ABC must also be 1, so C must be O reflected in AB (OC is perpendicular to AB).

The angle ACB is therefore 90°.

For ABD to be congruent to ABC, ABD must be the rotation of ABC about the midpoint of AB, and angle ADB is also 90°.

Thus ACBD is a rectangle, and since AB = $\sqrt{5}$, so is CD.

Questions 6-24 are the main questions, one a week-day.

Question 6 (Wednesday November 10th)

Below is a partly completed 4 by 4 magic square, where all rows, columns and both diagonals add to the same amount. What is x?

7			4
	10		
		x	
19			x+3

To get your final answer, multiply this by 4224.7 and take the integer part.

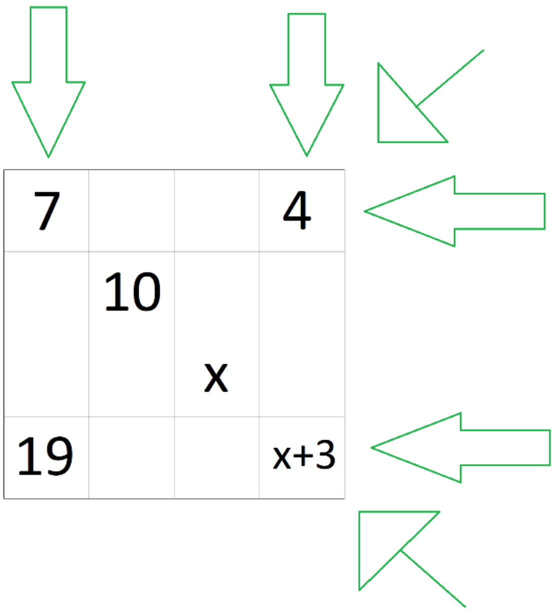
(54921)

Correct answer releases

y

Question 6 Answer

Suppose the row total is k.



If we add together the elements in these six directions, we get 6k.

This counts altogether

every element in the grid + 2(each corner element)

So 6k = 4k + 2(sum of corner elements) = 4k+ 2(33+x).

Which implies that k = 33 + x.

But k = 17 + 2x + 3 = 33 + x, so x = 13.

7	17	18	4
12	10	9	15
8	14	13	11
19	5	6	16

Note that with x = 13, a solution involving the consecutive integers from 4 to 19 is possible.

Question 7 (Thursday November 11th)

Can you work out what is happening here?

			160
a	b	c	180
d	e	f	2400
g	h	a	216
240	432	900	2304

The letters a to h are all positive integers.

What is the number a + b + c + d + e + f + g + h?

To get your final answer, multiply this by 483.4 and take the integer part.

(34804)

Correct answer releases

O

Question 7 Answer

In the below, a^k*b means

'a^k divides b and is the highest power of a that divides b'.

None of a, e, g, h or b are divisible by 5.

So 5^1 *d, 5^1 *f, 5^1 *c.

None of c, e or g are divisible by 3.

Thus 3^1 *a (from looking at the diagonal),

so 3^1 *b, 3^2*h, 3^1*f and 3 does not divide d.

For the powers of 2;

If 2^k*a, then 2^{8-2k}*e, but $k \leq 2$, $8 - 2k \leq 4$, so 2 = k.

So 2^2*a, 2^4*e, 2^1*g, 2^1*d while b, c, f, h are odd.

			160
12	3	5	180
10	16	15	2400
2	9	12	216
240	432	900	2304

12 + 3 + 5 + 10 + 16 + 15 + 2 + 9 = 72. Or else we can say

a divides into hcf(216, 180, 240, 2304, 900) = 12 so a = 1, 2, 3, 4, 6 or 12

then e = $2304/a^2$, and if e is to be a factor of 432, a = 12 and e = 16.

gh = 18 and dg = 20, so g = 1 or 2 but bh = 27 so h = 9 and g = 2.

The rest follows naturally.

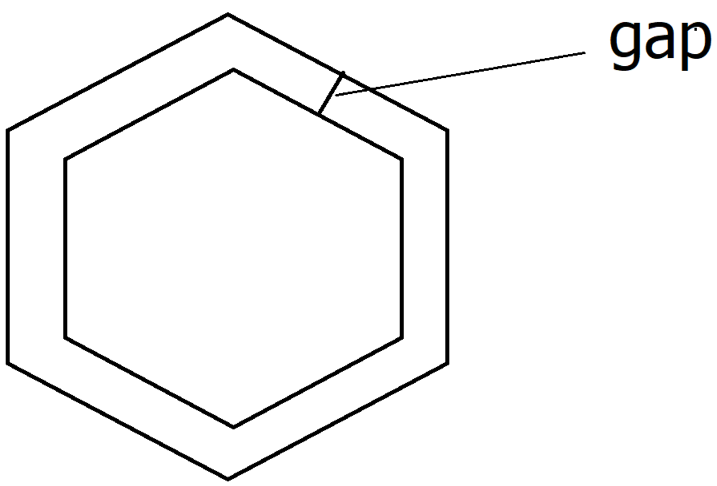
Question 8 (Friday November 12th)

A regular n-sided polygon ($3 \leq n \leq 1000$) has perimeter P.

A new regular n-sided shape with perimeter $P' = P + (1 - n/1000)^2$

is drawn symmetrically around the first.

For what value of n is the gap (as illustrated below for n = 6) biggest?



You may need to use a graph-plotting program

to solve an equation approximately here.

To get your final answer, multiply this by 4541.62 and take the integer part.

(68124)

Correct answer releases

t

Question 8 Answer

Suppose the first n-sided regular polygon

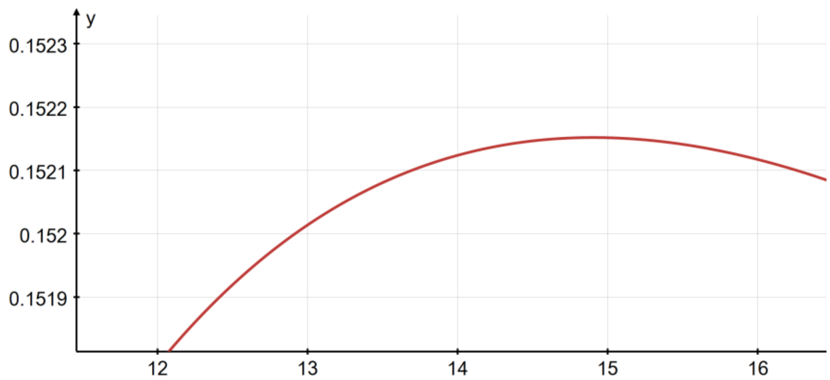
can be inscribed in a circle of radius r.

$$P = 2nr\sin\frac{\pi}{n}\ .$$

So $P' = 2nr\sin\frac{\pi}{n} + \left(1-\frac{n}{1000}\right)^2 = 2nR\sin\frac{\pi}{n} \Rightarrow R - r = \frac{\left(1-\frac{n}{1000}\right)^2}{2n\sin\frac{\pi}{n}}.$

Gap = $P' = \frac{\left(1-\frac{n}{1000}\right)^2}{2n\sin\frac{\pi}{n}}\cos\frac{\pi}{n} = \frac{\left(1-\frac{n}{1000}\right)^2}{2n\tan\frac{\pi}{n}}.$

Plotting the graph of $y = \frac{\left(1-\frac{x}{1000}\right)^2}{2x\tan\frac{\pi}{x}}$ gives a maximum close to n = 15.



Alternative solution; assuming the question is well-posed,

the answer must be independent of P. So choose P = 0.

If we have gap g and side length s), $\tan\left(\frac{\pi}{n}\right) = \frac{s}{2g}, ns = \left(1-\frac{n}{1000}\right)^2$.

This leads to the above graph.

Question 9 (Monday November 15th)

You are given that the infinite series

$$\cos^2 x - \sin^2 x + \cos^4 x - \sin^4 x + \cos^6 x - \sin^6 x + \ldots$$

adds to 1 for some value of x.

You are also given that $0 < x < \pi/2$.

What to 3 s.f. is x?

To get your final answer, multiply this by 35489 and take the integer part.

(23635)

Correct answer releases

j

Question 9 Answer

$$\cos^2 x - \sin^2 x + \cos^4 x - \sin^4 x + \cos^6 x - \sin^6 x + \ldots$$

$$= \cos^2 x + \cos^4 x + \cos^6 x + \ldots . - (\sin^2 x + \sin^4 x + \sin^6 x + \ldots)$$

$$= \frac{\cos^2 x}{1 - \cos^2 x} - \frac{\sin^2 x}{1 - \sin^2 x}$$

$$= \frac{\cos^4 x - \sin^4 x}{\cos^2 x \sin^2 x} = \frac{(\cos^2 x - \sin^2 x)\left(\cos^2 x + \sin^2 x\right)}{\cos^2 x \sin^2 x} = \frac{4\cos(2x)}{\sin^2(2x)} = 1$$

Let $\cos(2x) = c.$

So $1 - c^2 = 4c \Rightarrow c^2 + 4c - 1 = 0 \Rightarrow \cos(2x) = \dfrac{-4 + \sqrt{20}}{2} = \sqrt{5} - 2$

$\Rightarrow x = 0.666(3s.f.)$

Or alternatively

$$\frac{\cos^4 x - \sin^4 x}{\cos^2 x \sin^2 x} = 1 \Rightarrow \frac{1 - \tan^4 x}{\tan^2 x} = 1,$$

which is a quadratic in tan x that leads to the same answer.

Question 10 (Tuesday November 16th)

The sequence $u_1, u_2, u_3\dots$ can be written as (u_n).

You are given that

- (u_n) is an arithmetic progression

with first term a and common difference b

- (v_n) is an arithmetic progression

with first term c and common difference d

- $(u_n + v_n)$ is an arithmetic progression

with first term 5 and common difference 8.

- $(u_n v_n)$ is $(w_n + k(n - 1)^2)$, where (w_n) is an arithmetic progression

with first term 6 and common difference 7,

and where k is a constant.

Find $|a\times b\times c\times d|$.

To get your final answer, multiply this by 12.2 and take the integer part.

(11199)

Correct answer releases

y

Question 10 Answer

$u_n + v_n = a + (n-1) b + c + (n-1)d$

$=(a + c) + (n - 1)(b + d).$

So $a + c = 5$, $b + d = 8$.

$u_n v_n = (a + (n - 1) b)(c + (n - 1)d)$

$= ac + (n - 1)(ad + bc) + (n - 1)^2 bd.$

Thus $ac = 6$, $(ad + bc) = 7$.

Thus $(a, b, c, d) = (3, 17, 2, -9)$ or $(2, -9, 3, 17)$, and so $|a\times b\times c\times d| = 918$.

Question 11 (Wednesday November 17th)

In a cyclic quadrilateral ABCD,

the lengths of the sides AB, BC, CD and DA

are in arithmetic progression with common difference $d > 0$.

If angle DAB is 60^o, what (to 3s.f.) is AB/d?

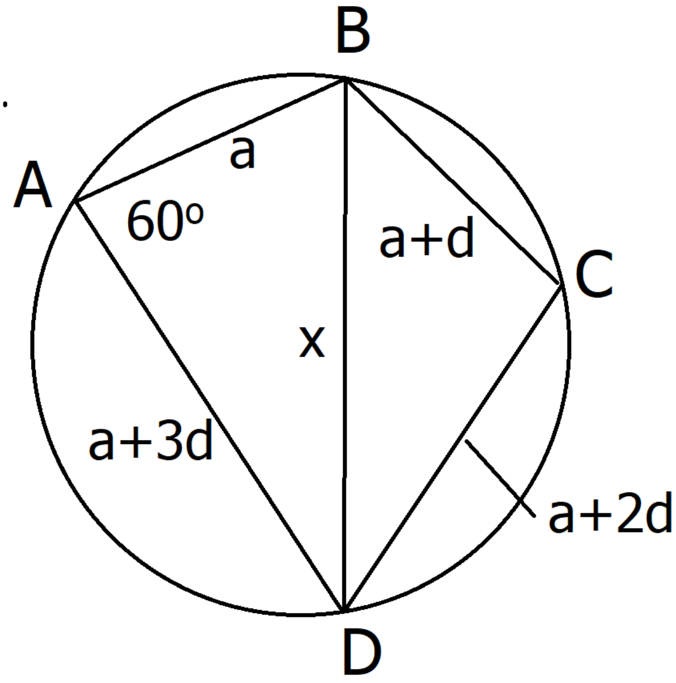
To get your final answer, multiply this by 63545 and take the integer part.

(19254)

Correct answer releases

t

Question 11 Answer



Let AB/d = k.

Using the cosine rule;

$x^2 = a^2 + (a + 3d)^2 - 2a(a + 3d)0.5 = (a + d)^2 + (a + 2d)^2 + 2(a + d)(a + 2d)0.5$

$\Rightarrow 2a^2 + 6ad + 9d^2 - a^2 - 3ad = 2a^2 + 6ad + d^2 + 4d^2 + a^2 + 3ad + 2d^2$

$\Rightarrow 2d^2 = 2a^2 + 6ad \Rightarrow k^2 + 3k - 1 = 0 \Rightarrow k = \frac{-3 + \sqrt{9 + 4}}{2} = 0.303(3s.f.)$

Question 12 (Thursday November 18th)

You are given the cubic polynomial $y = x^3 + ax^2 + bx + c$.

Let $\frac{d^n y}{dx^n}(k)$ denote the nth derivative of y with respect to x

when evaluated at $x = k$.

You are given that $\frac{d^n y}{dx^n}(m) = \frac{d^m y}{dx^m}(n)$ for all integer m, n,

where $0 \leq m \leq 2$, $0 \leq n \leq 2$.

What is $a \times b \times c$?

To get your final answer, multiply this by 9536 and take the integer part.

(28254)

Correct answer releases

y

Question 12 Solution

We have the three equations

y(1) = y’(0), y(2) = y’’(0), y’(2) = y’’(1).

Thus

$$a + b + c + 1 = b$$

$$4 \cdot a + 2 \cdot b + c + 8 = 2 \cdot a$$

$$4 \cdot a + b + 12 = 2 \cdot a + 6$$

The solution to these three simultaneous equations is

$$c = \frac{2}{3}$$

$$b = -\frac{8}{3}$$

$$a = -\frac{5}{3}$$

Which gives abc = 80/27.

Question 13 (Friday November 19th)

Fit the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 into the squares below

in a way that makes the equations truthful.

$$\mathrm{X} =$$

$$a^{e} \times b \times c \times d \times fg$$

$$= h^{3} \times i \times j \times 112$$

where

- fg is a two digit number,
- $1 < a < b < c < d < e$ and
- $1 < h < i < j$.

What is X?

To get your final answer, multiply this by 0.22 and take the integer part.

(70963)

Correct answer releases

h

Question 13 Solution

$112 = 2^4 \times 7$,

so 7 must be part of the top equation.

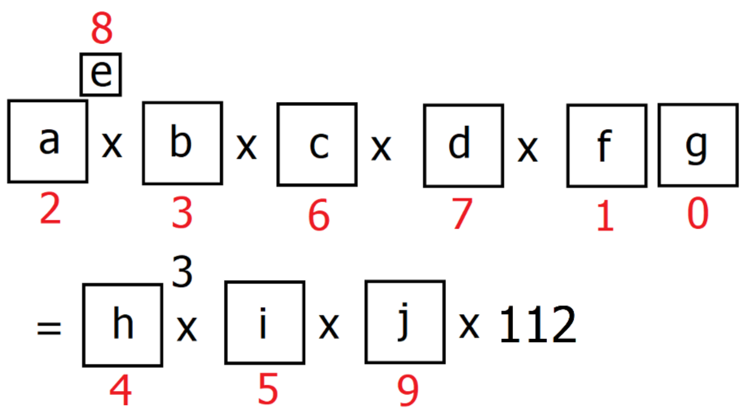
The only place the 1 can go is for f, while g is 0.

5 must divide both sides, so 5 is in the bottom equation.

a and h must be 2, 3 or 4, but 3 doesn't work for either.

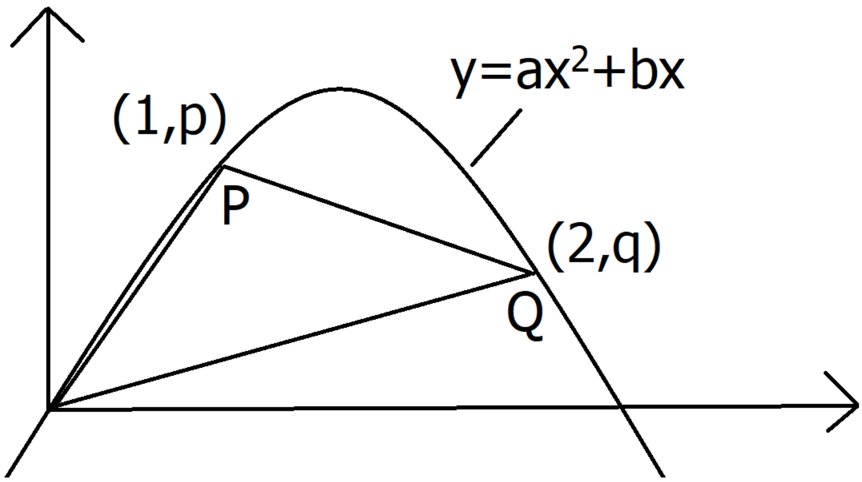
We must have 9 on one side of the equation, and 3 and 6 on the other.

a has to be 2, so e has to be 8, and h is 4.



x = 322560.

Question 14 (Monday November 22nd)



P = (1, p) and Q = (2, q) lie on the parabola $y = ax^2 + bx$.

The area of triangle OPQ

is half the area enclosed by the curve and the x-axis.

What is |a/b| to 3s.f.?

To get your final answer, multiply this by 62154 and take the integer part.

(27161)

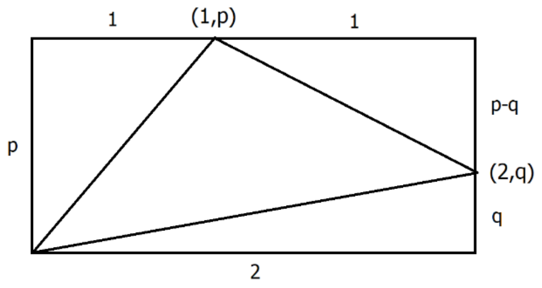
Correct answer releases

g

Question 14 Answer

p = a + b, q = 4a + 2b. Say area of OPQ = A.

We can draw a rectangle enclosing the triangle.



Not to scale

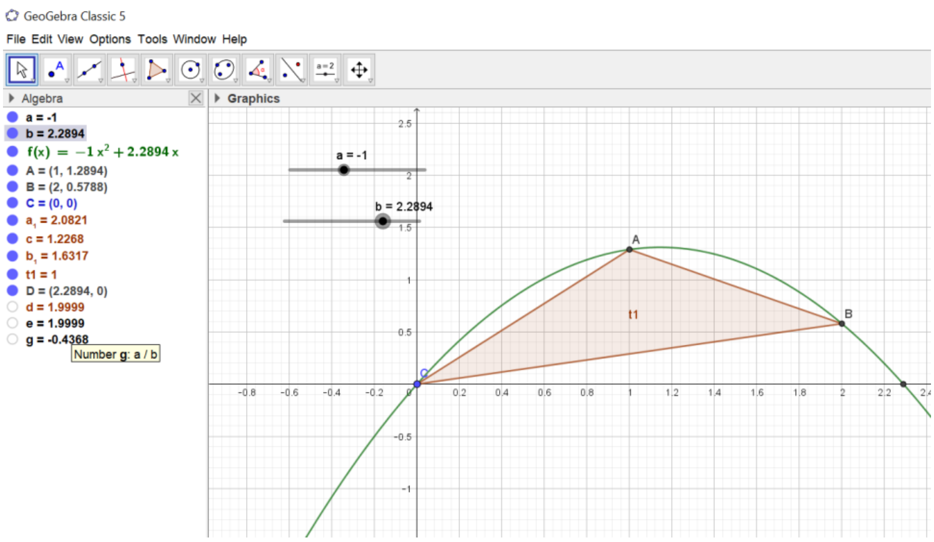
A = 2p – p/2 – (p - q)/2 – q = p – q/2.

Integrating, $\int_0^{-b/a} ax^2 + bxdx = \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^{-b/a} = \frac{b^3}{6a^2}$.

Thus $\frac{b^3}{12a^2} = (a+b) - (2a+b) = -a \Rightarrow \frac{a}{b} = -12^{-1/3}$.

Thus $|\frac{a}{b}| = 0.437(3s.f.)$

Or else there is a Geogebra approach you could take.



Question 15 (Tuesday November 23rd)

Given a triangle T, with sides a, b and c and perimeter P,

where $a, b, c > \frac{P}{6}$,

define D(T), the dual of T, to be the triangle with sides

$$\frac{2}{3}P - a, \frac{2}{3}P - b, \frac{2}{3}P - c.$$

Notice that T and D(T) have the same perimeter, and that D(D(T)) = T.

(Why is the condition $a, b, c > \frac{P}{6}$ important?)

If a = 7, b = 11, then what is the smallest possible value for c

if the areas of T and D(T) are equal?

Note; you may find Heron's formula for the area of a triangle useful.

To get your final answer, multiply this by 1245.6 and take the integer part.

(11210)

Correct answer releases

h

Question 15 Solution

Wikipedia;

Heron's formula states that the area of a triangle whose sides have lengths a, b, and c is

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semi-perimeter of the triangle; that is,

$$s = \frac{a+b+c}{2}.$$ [2]

For T, P = 18 + c, s = 9 + c/2.

Thus $A = \sqrt{\left(9+\frac{c}{2}\right)\left(\frac{c}{2}+2\right)\left(\frac{c}{2}-2\right)\left(9-\frac{c}{2}\right)}$.

For D(T), P = 18 + c, s = 9 + c/2.

Thus $A = \sqrt{\left(9+\frac{c}{2}\right)\left(4-\frac{c}{6}\right)\left(8-\frac{c}{6}\right)\left(\frac{5c}{6}-3\right)}$.

Equating the two expressions for A gives us

(24 - c)(5c - 18)(48 - c) = 27(c + 4)(c - 4)(18 - c).

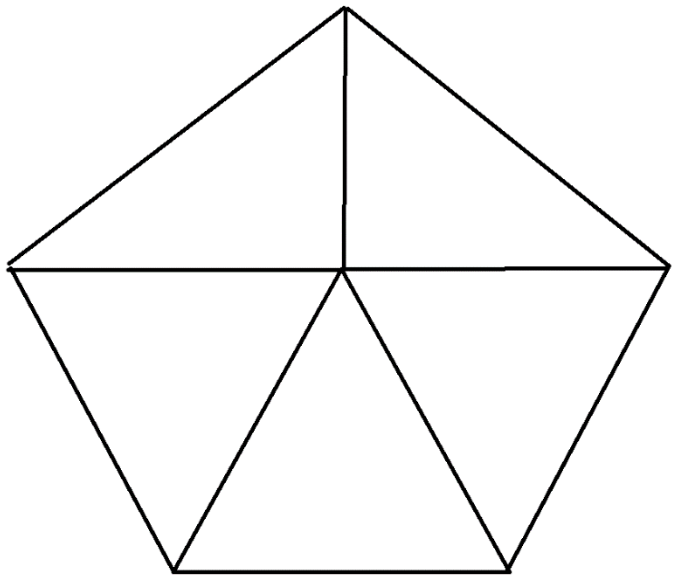
Solving for c gives the values 3, 9, and 15.

The value 3 doesn’t give a viable triangle for T.

So the two possible values for c are 9 and 15,

giving us c = 9 as our answer.

Question 16 (Wednesday November 24th)



The diagram shows three congruent equilateral triangles

beneath two congruent right-angle triangles

creating an irregular pentagon of area A.

If the areas of the five triangles are all equal,

and the pentagon has perimeter P, find P^2/A to 3s.f.

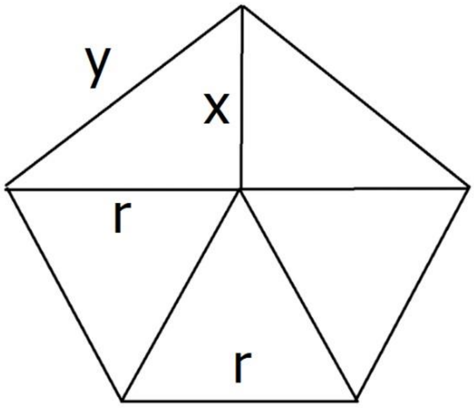
To get your final answer, multiply this by 1842 and take the integer part.

(27077)

Correct answer releases

W

Question 16 Solution



Let the side of one of the equilateral triangles be r.

Then its area will be $\frac{1}{2}r\left(\frac{\sqrt{3}r}{2}\right)=\frac{\sqrt{3}r^2}{4}$.

This means that $\frac{1}{2}rx=\frac{1}{4}\sqrt{3}r^2 \Rightarrow x=\frac{1}{2}\sqrt{3}r$.

Thus $y^2=x^2+r^2=\frac{3r^2}{4}+r^2 \Rightarrow y=\frac{\sqrt{7}}{2}r.$

This gives

$$\frac{P^2}{A}=\frac{(3r+2y)^2}{\left(5\frac{\sqrt{3}}{4}r^2\right)}=\frac{9r^2+12ry+4y^2}{\left(5\frac{\sqrt{3}}{4}r^2\right)}$$

$$=\frac{9+12\frac{\sqrt{7}}{2}+4\frac{7}{4}}{5\frac{\sqrt{3}}{4}}=\frac{16+6\sqrt{7}}{1.25\sqrt{3}}=14.7(3s.f.)$$

Question 17 (Thursday November 25th)

1

1 1

1 2 1

1 3 3 1

...

1 n ... n 1

The top n + 1 rows of Pascal's Triangle are shown above.

You are told that

the average of all these numbers is 1533.

What is n?

You could find Excel useful here.

To get your final answer, multiply this by 4124.7 and take the integer part.

(70119)

Correct answer releases

C

Question 17 Solution

The sum of the numbers in the nth row

(taking the top number 1 as row 0) of Pascal's Triangle is 2^n.

(Try expanding $(1+1)^n$ using the Binomial Theorem).

So the sum of all the numbers shown is

$$2^0+2^1+\ldots+2^n = \frac{1\left(2^{n+1}-1\right)}{2-1} = 2^{n+1}-1 \ .$$

The number of numbers is

$$1 + 2 + 3+\ldots + (n+1) = \frac{(n+1)(n+2)}{2} \ .$$

So the average we seek is $\frac{2(2^{n+1}-1)}{(n+1)(n+2)}$. Using Excel gives this;

	A	B
1	1	1
2	2	1.166667
3	3	1.5
4	4	2.066667
5	5	3
6	6	4.535714
7	7	7.083333
8	8	11.35556
9	9	18.6
10	10	31.01515
11	11	52.5
12	12	90.01099
13	13	156.0286
14	14	273.0583
15	15	481.875
16	16	856.6732
17	17	1533
18	18	2759.405

So n is 17.

Question 18 (Friday November 26th)

Define a rectangle to be DAPPER

if its diagonal, area and perimeter

are all positive integer values.

If a rectangle has perimeter 48 and a diagonal of length 20,

show that it is dapper, and find its area.

To get your final answer, multiply this by 263.9 and take the integer part.

(23223)

Correct answer releases

p

Question 18 Solution

Say that the rectangle has sides x and y.

We have that 48 = 2x + 2y, so 24 = x + y.

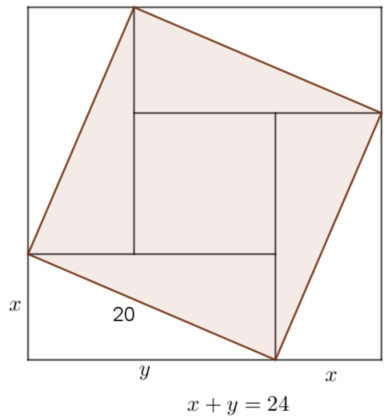
We also have that $20^2 = x^2 + y^2$, by Pythagoras.

$24^2 = 576 = (x + y)^2 = x^2 + 2xy + y^2 = 400 + 2xy$.

Thus 176 = 2xy, and xy = area = 88.

Thus the rectangle is dapper, and we have its area.

Or else, this diagram gives this to us;



$AreaRectangle : \frac{24^2 - 20^2}{2}$

More generally…

any rectangle where the diagonal d and the semi-perimeter s = p/2,

are both even, or where d and s are both odd, is DAPPER.

Let one of the sides x, so the other is (s-x).

Then $d^2 = (s-x)^2 + x^2 = 2x^2 - 2sx + s^2 \Rightarrow x^2 - sx + \frac{(s^2-d^2)}{2} = 0.$

The roots of this equation give the lengths of the sides

and the product of those roots, which is the area of the rectangle, is $\frac{(s^2-d^2)}{2}$.

This will be an integer if s and d are either both even or both odd.

Question 19 (Monday November29th)

How many four-digit numbers (in base 10) are there so that

- all the digits are different
- two of the digits add to the sum of the other two?

A number cannot begin with 0.

To get your final answer, multiply this by 26.3 and take the integer part.

(29035)

Correct answer releases

i

Question 19 Solution

The pair (96)(87) leads to 24 different numbers (= 4!).

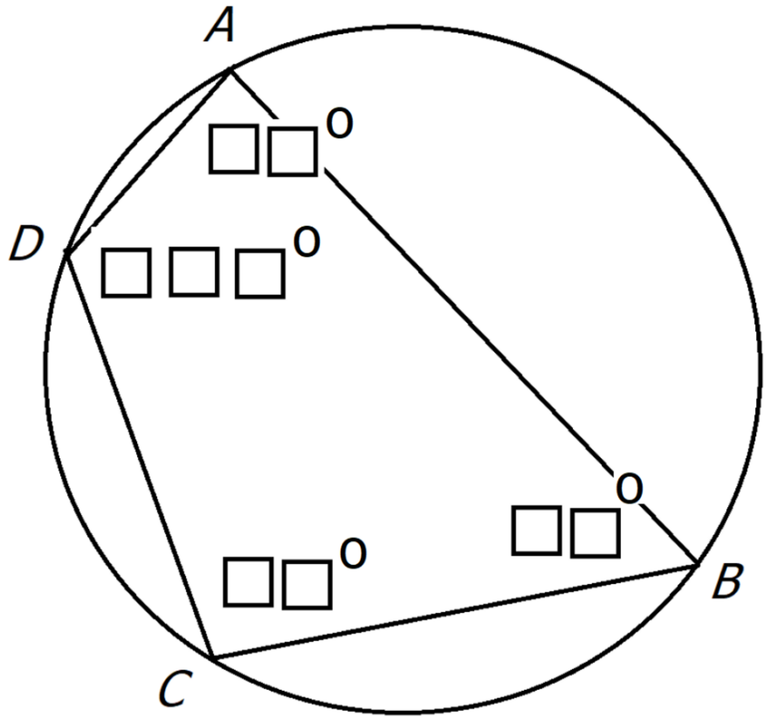
The pair (30)(21) leads to 18 different numbers (= 3×3!).

The table below gives the possibilities.

Sum		24	18	Total
3	(30)(21)	0	1	18
4	(40)(31)	0	1	18
5	(50)(41)(32)	1	2	60
6	(60)(51)(42)	1	2	60
7	(70)(61)(52)(43)	3	3	126
8	(80)(71)(62)(53)	3	3	126
9	(90)(81)(72)(63)(54)	6	4	216
10	(91)(82)(73)(64)	6	0	144
11	(92)(83)(74)(65)	6	0	144
12	(93)(84)(75)	3	0	72
13	(94)(85)(76)	3	0	72
14	(95)(86)	1	0	24
15	(96)(87)	1	0	24
				1104

The answer is 1104.

Question 20 (Tuesday November 30th)



The above diagram is not to scale.

The points A, B, C and D form a cyclic quadrilateral.

The circle ABCD is shown in the diagram.

The angles at A and B are acute, the angles at C and D are obtuse.

The digits 1 to 9 are put into the squares to make a truthful picture.

How many different ways is it possible to do this?

To get your final answer, multiply this by 2389.1 and take the integer part.

(38225)

Correct answer releases

g

Question 20 Solution

The first digit at D must be a 1.

The angles at A and C add to 180^o, so the first digits here must be 8 and 9.

So C begins with a 9, A begins with an 8 (since A is acute).

The second digits at A and C must be (3, 7) or (4, 6)

The final digits at D and B must be (3, 7) or (4, 6).

So the second digit for D must be 2 or 5, and the same is true for the angle at B.

So (AA, BB, CCC, DD) could be

(83,54,97,126) (87,54,93,126) (84,53,96,127) (86,53,94,127)

(83,56,97,124) (87,56,93,124) (84,57,96,123) (86,57,94,123)

(83,24,97,156) (87,24,93,156) (84,23,96,157) (86,23,94,157)

(83,26,97,154) (87,26,93,154) (84,27,96,153) (86,27,94,153)

There are 16 possibilities.

Question 21 (Wednesday December 1st)

Define $$f(x) = \int_0^x \left(\int \left(\int y - k dy \right) dy \right) dy$$,

where k is a constant.

If f(1) = 1, f(2) = 2, and f(3) = 3, what is k?

To get your final answer, multiply this by 62349 and take the integer part.

(93523)

Correct answer releases

h

Question 21 Answer

Integrating once, we get $\frac{y^2}{2} - ky + a$,

integrating twice we get $\frac{y^3}{6} - \frac{ky^2}{2} + ax + b$,

integrating a third time we get $\frac{y^4}{24} - \frac{ky^3}{6} + \frac{ay^2}{2} + bx + c$.

For the limits 0 and 1, we get $\frac{1}{24} - \frac{k}{6} + \frac{a}{2} + b = 1 \Rightarrow -4k + 12a + 24b = 23$,

For the limits 0 and 2, we get $\frac{16}{24} - \frac{8k}{6} + \frac{4a}{2} + 2b = 2 \Rightarrow -2k + 3a + 3b = 2$,

For the limits 0 and 3, we get $\frac{81}{24} - \frac{27k}{6} + \frac{9a}{2} + 3b = 3 \Rightarrow -12k + 12a + 8b = -1$.

We now have three simultaneous equations in three unknowns.

These solve to give $k = \frac{3}{2}, a = \frac{11}{12}, b = \frac{3}{4}$, so the answer is 1.5.

Question 22 (Thursday December 2nd)

Define L(S) to be the longest straight line segment

lying wholly inside the shape S.

So if T is a triangle, L(T) is the longest side,

and if R is a rectangle, L(R) is the length of the diagonal.

A convex pentagon P = ABCDE has right angles at A and B,

and AB = x, BC = 1, CD = 1, DE = 1, and EA = 1, where $0 < x < 2$.

For what value of x (to 3.s.f.) is L(P) a minimum?

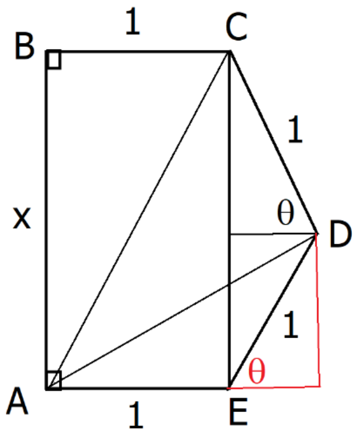
To get your final answer, multiply this by 42247 and take the integer part.

(64215)

Correct answer releases

g

Question 22 Solution

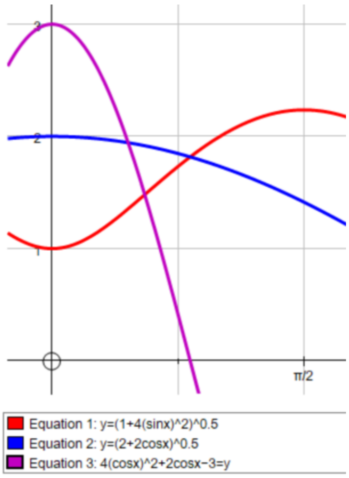


There are two candidates for the longest line segment in P, AC and AD.

$\sin\theta = \frac{x}{2}$, AC = $\sqrt{1+x^2} = \sqrt{1+4\sin^2\theta}$, AD = $\sqrt{(1+\cos\theta)^2+\sin^2\theta} = \sqrt{2+2\cos\theta}$.

We can draw the graphs of AC against θ (red), and AD against θ (blue)

(note we need to swap θ for x in the equations below).



We can see the minimum value for (the larger of AC and AD)

occurs when AC = AD. So we seek the solution to

$$1+4\sin^2\theta = 2+2\cos\theta \Rightarrow 4\cos^2\theta + 2\cos\theta - 3 = 0.$$

(see the purple curve above). This gives

$$\cos\theta = \frac{-2+\sqrt{4+48}}{8} \Rightarrow \theta = 0.86138...$$
$$\Rightarrow x = 1.52(3s.f.)$$

Question 23 (Friday December 3rd)

A sequence is defined as follows for n ≥ 3;

$$u_1 = a, u_2 = b, u_n = \frac{u_{n-1} + u_{n-2}}{2}.$$

So the next term is the average of the previous two.

You are told that as n tends to infinity, u_n tends to 12.

Consider the sequence with the same rule that starts

$$u_1 = b, u_2 = c \ ;$$

you are told that as n tends to infinity here, u_n tends to 15.

Consider the sequence with the same rule that starts

$$u_1 = c, u_2 = a \ ;$$

you are told that as n tends to infinity here, u_n tends to 18.

What is a + b + c?

To get your final answer, multiply this by 1234.5 and take the integer part.

(55552)

Correct answer releases

h

Question 23 Solution

The sequence goes

$$a, b, \frac{a+b}{2}, \frac{a+3b}{4}, \frac{3a+5b}{8}, \frac{5a+11b}{16}, \frac{11a+21b}{32}, \frac{21a+43b}{64}, \frac{43a+85b}{128}\ldots$$

which is

$$a, b, \frac{a+b}{2}, \frac{a+(2+1)b}{3+1}, \frac{3a+(6-1)b}{9-1}, \frac{5a+(10+1)b}{15+1}, \frac{11a+(22-1)b}{33-1}, \frac{21a+(42+1)b}{63+1}, \frac{43a+(86-1)b}{129-1}\ldots$$

If we ignore the 1s in the above (and as the numbers get bigger, the 1s do become more and more insignificant in the calculation) then we get $\frac{a+2b}{3}$ each time.

So we can say that if $u_1 = a, u_2 = b, u_n = \frac{u_{n-1}+u_{n-2}}{2}$ for $n \geq 3$,

u_n gets closer and closer to $\frac{a+2b}{3}$ as n gets larger and larger.

Alternatively, we can write the sequence as

$$a, a+(b-a), a+(b-a)\left(1-\frac{1}{2}\right), a+(b-a)\left(1-\frac{1}{2}+\frac{1}{4}\right), a+(b-a)\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}\right)\ldots$$

and so u_n tends to

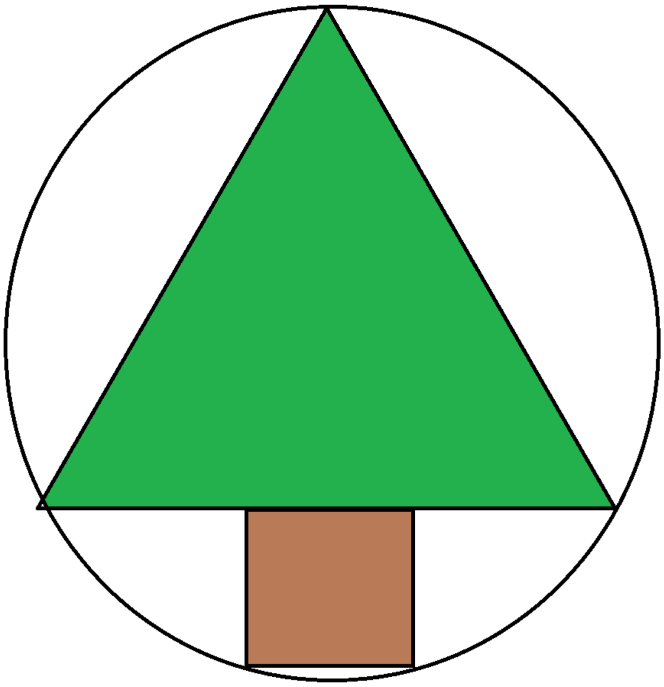
$$a+\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots\right)(b-a) = a+\frac{2}{3}(b-a) = \frac{a+2b}{3}.$$

So we have $\frac{a+2b}{3} = 12, \frac{b+2c}{3} = 15, \frac{c+2a}{3} = 18.$ Adding these gives a + b + c = 45.

Call the three sequences u_n, v_n, w_n. Form a fourth sequence with terms $u_n+v_n+w_n$.

This term starts a + b + c, a + b + c, … and by construction all terms in this new sequence are equal and so must be 12 + 15 + 18 = 45.

Question 24 (Monday December 6th)



A Christmas tree consists of a green equilateral triangle and a brown square inside a circle of radius 1 as shown.

What is the area of the tree?

Give your answer to 3s.f.

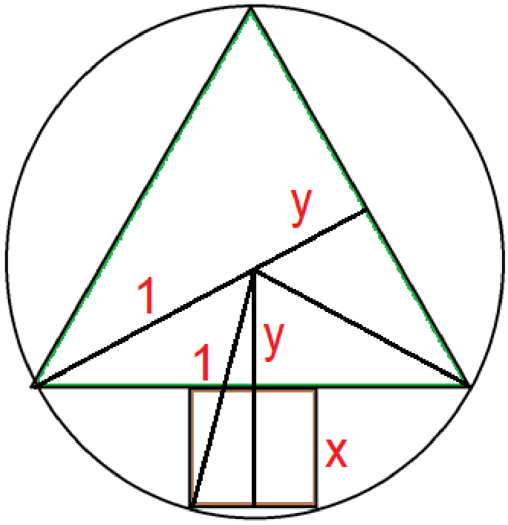
To get your final answer, multiply this by 11111 and take the integer part.

(16888)

Correct answer releases

W

Question 24 Solution



Side of triangle = $2\sin 60^o = \sqrt{3}.$

$$1 + y = \sqrt{3}\sin 60^o \Rightarrow y = \frac{1}{2} \ .$$

$$\left(\frac{x}{2}\right)^2 + (x+y)^2 = 1 \Rightarrow \frac{x^2}{4} + x^2 + x + \frac{1}{4} = 1 \Rightarrow 5x^2 + 4x - 3 = 0$$

$$\Rightarrow x = \frac{-4 + \sqrt{16+60}}{10} \Rightarrow x = 0.471779...$$

Area of the triangle = $\frac{1}{2}ab\sin\theta = \frac{1}{2}\sqrt{3} \times \sqrt{3} \times \frac{\sqrt{3}}{2} = 1.299038...$

Thus the total area of the tree to 3s.f. is 1.52.

Ritangle 2021, Final Task (Tuesday December 4th, 4pm)

Stage 2

Well done on reaching Stage 2 of Ritangle 2021.

You've still got some way to go…

You need to solve these next eight questions.

You may need to pick up some new maths along the way –

modular arithmetic, and matrix algebra.

We hope this stands you in good stead for the future!

Questions 28-32 could be especially helpful for the final Stage 2 task.

Remember, you are allowed to ask your teachers or others

for help over general theory,

but not for help on the particular problem you are tackling.

If your answer to each of the eight questions checks out correctly,

you will be given a clue (eight in total).

Following the eight questions is one last clue

towards solving Stage Two completely.

If you answer this final task correctly,

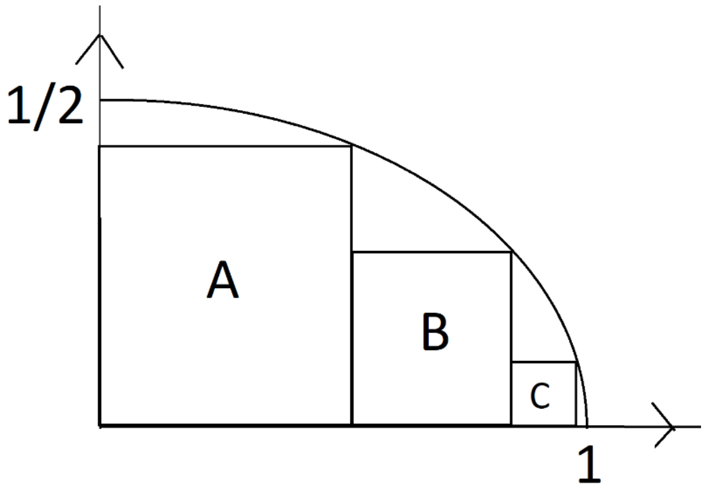
you will have a sixteen-character string that will unlock Stage 3.

Let us give you a free clue to start you off!

Start

⇩

Question 25



The diagram shows the part of the ellipse $x^2 + 4y^2 = 1$

that lies in the first quadrant.

The area between the ellipse, the x-axis and the y-axis in the diagram is E.

A, B and C are the first three squares

in an infinite sequence of squares.

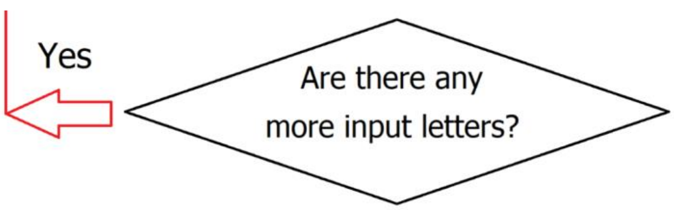
Let S be the sum of the areas of the squares A, B, C… to infinity.

What (to 4s.f.) is S/E?

To get your final answer, multiply this by 52469 and take the integer part.

(44782)

Correct answer releases



Question 25 Answer

Let the sides of the squares be a, b, c, …

The area of the ellipse $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$ is πpq, and here p = 1, q = 0.5.

So the area of the part-ellipse in the question is π/8.

Thus we are trying to find $\frac{a^2+b^2+c^2+\dots}{\pi/8}$ to 4s.f.

The top right corner of each square is on the ellipse,

so its coordinates are (a + b + c… + x, x).

Let u_1 = a, u_2 = a + b, u_3 = a + b + c, …

so (a + b + c… + x, x) = (u_n + x, x). Thus

$$(u_n+x)^2+4x^2=1 \Rightarrow 5x^2+x(2u_n)+(u_n{}^2-1)=0 \Rightarrow x=-\frac{u_n}{5}+\frac{\sqrt{5-4u_n{}^2}}{5}$$

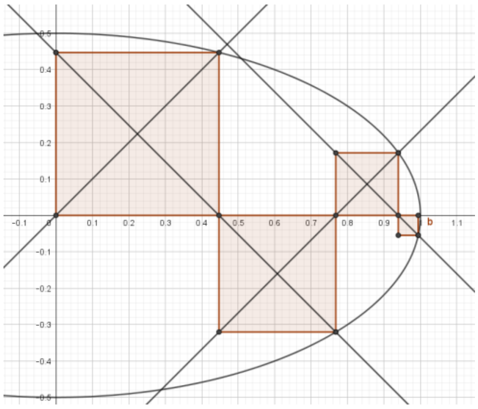
But x = u_{n+1} - u_n, so $u_{n+1}=\frac{4u_n}{5}+\frac{\sqrt{5-4u_n{}^2}}{5}$.

It is easy to see from the ellipse's equation that a = u_1 = $\frac{1}{\sqrt{5}}$. Now using Excel;

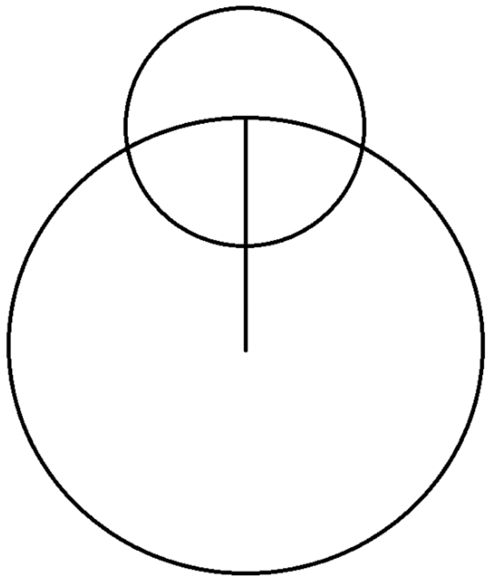
n	u_n	a,b,c,…	a^2,b^2,c^2…	adding up
1	0.447213595	0.447213595	0.2	0.2
2	0.767648907	0.320435312	0.102678789	0.302678789
3	0.939256673	0.171607766	0.029449225	0.332128014
4	0.993990384	0.054733711	0.002995779	0.335123793
5	0.999929452	0.005939068	3.52725E-05	0.335159066
6	0.99999999	7.05376E-05	4.97555E-09	0.335159071
7	1	9.95111E-09	9.90245E-17	0.335159071
8	1	0	0	0.335159071
9	1	0	0	0.335159071
10	1	0	0	0.335159071

$$\frac{a^2+b^2+c^2+\dots}{\pi/8}=0.8535(4s.f.)$$

Alternatively, this diagram is a good one to work with.



Question 26



A circle of radius 2 is intersected by a circle of radius 1,

where the centre of the smaller circle lies on the larger.

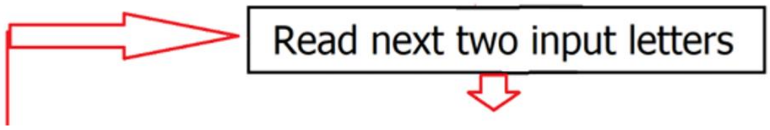
What percentage of the area of the smaller circle is inside the larger?

Give your answer to 3s.f.

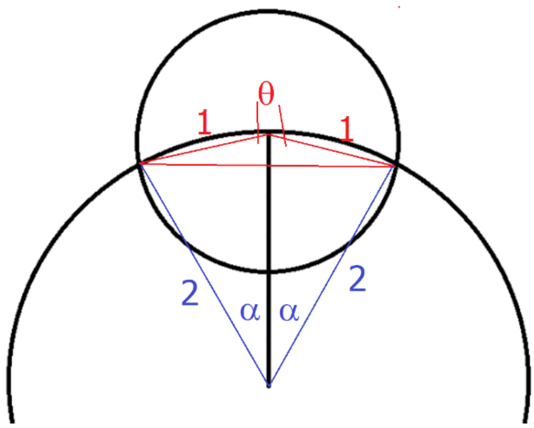
To get your final answer, multiply this by 1234 and take the integer part.

(55159)

Correct answer releases

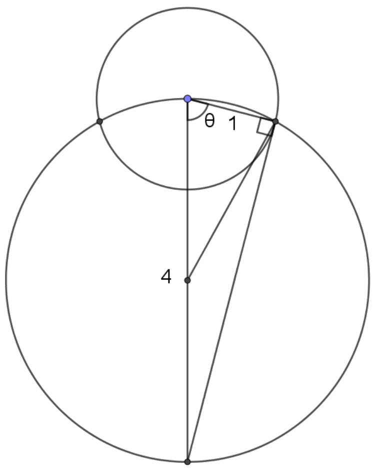


Question 26 Answer



Area of overlap is

$$\left(\frac{1}{2}2^2 2\alpha - \frac{1}{2}2^2\sin(2\alpha)\right) + \left(\frac{1}{2}1^2 2\theta - \frac{1}{2}1^2\sin(2\theta)\right)$$
$$= 4\alpha + \theta - 2\sin(2\alpha) - \frac{\sin(2\theta)}{2}.$$



Since the angle in a semicircle is 90°, we have $\cos\theta = 0.25$.

$$\cos\theta = 0.25 \Rightarrow \theta = 1.318116...$$
$$2\theta + \alpha = \pi \Rightarrow \alpha = 0.5053605...$$

Substituting these values in, we find that the area of the overlap is

1.403066…

Thus the percentage of the smaller circle inside the larger is

44.7%(3s.f.)

Question 27

Rishi needs £1 trn (one trillion pounds = one million million pounds) to repair his house.

He has the opportunity to invest 1p with the Leprechaun Unlimited Corporation (LUC).

The investment has the following features:

- with a probability of 0.98, the money doubles overnight,
- with a probability of 0.01 the money is returned unchanged the next day, and
- with a probability of 0.01 everything is lost, and Rishi cannot restart.

Unless the money is all lost, Rishi plans to continue to entrust the accumulated sum

to LUC on each subsequent night, with the same probabilities.

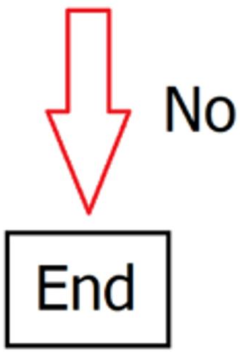
What is the probability of Rishi achieving his target of £1 trn within three years?

Express your answer as a decimal, to three significant figures.

To get your final answer, multiply this by 87654 and take the integer part.

(54433)

Correct answer releases



Solution to Question 27

We are asked to examine the cases where 'everything is lost' does not happen.

How many doublings are required to reach 1 trillion pounds?

1 trillion is 10^{12}, and $0.01 \times 10^{14} = 10^{12}$.

$2^n = 10^{14} \Rightarrow n\log 2 = 14 \Rightarrow n = 46.5...$ so 47 doublings are required.

Check; $2^{46} = 7.03\times10^{13}$, $2^{47} = 1.41\times10^{14}$.

These could be combined with as many 'no change' events as we like,

but the final event must be a doubling.

So adding up, the probability of success is

$$0.98^{47} + {}^{47}C_1(0.98^{47})(0.01) + {}^{48}C_2(0.98^{47})(0.01)^2$$

$$+ {}^{49}C_3(0.98^{47})(0.01)^3 + {}^{50}C_4(0.98^{47})(0.01)^4 \ldots$$

Or best strategy is to add these up using Excel.

	A	B	C	D	E	F
1	***Count***	***(0.98)^47***	***(0.01)^n***	***(n+46)C(n)***	***Product***	***Sum***
2	0	0.386924	1	1	0.386924	0.386924
3	1	0.386924	0.01	47	0.181854	0.568778
4	2	0.386924	0.0001	1128	0.043645	0.612423
5	3	0.386924	0.000001	18424	0.007129	0.619552
6	4	0.386924	1E-08	230300	0.000891	0.620443
7	5	0.386924	1E-10	2349060	9.09E-05	0.620534
8	6	0.386924	1E-12	20358520	7.88E-06	0.620542
9	7	0.386924	1E-14	1.54E+08	5.96E-07	0.620542
10	8	0.386924	1E-16	1.04E+09	4.03E-08	0.620542
11	9	0.386924	1E-18	6.36E+09	2.46E-09	0.620542
12	10	0.386924	1E-20	3.56E+10	1.38E-10	0.620542
13	11	0.386924	1E-22	1.85E+11	7.14E-12	0.620542

So our answer is 0.621 (3s.f.)

Question 28

If ‘^’ means ‘raise to the power of’,

so that 5^7 means 5^7,

what are the last five digits of 5^(7^(5^7))?

Take these final five digits as your final answer.

(78125)

Correct answer releases

Print out these two letters

Solution to Question 28

$5^1 = 5$, $5^2 = 25$, $5^3 = 125$, $5^4 = 625$ $5^5 = 3125$, $5^6 = 15625$,

$5^7 = 78125$, $5^8 = 390625$, $5^9 = 1953125$, $5^{10} = \ldots 65625$,

$5^{11} = \ldots 28125$, $5^{12} = \ldots\ldots 40625$, $5^{13} = \ldots 03125$,

$5^{14} = \ldots 15625$, the same final five digits as 5^6.

So multiplying 5^n by 5^8 does not change the last 5 digits if $n > 5$.

So what is $7^{(5^7)} \pmod 8$?

$7^{(5^7)} \pmod 8 = (-1)^{78125} \pmod 8$

$= -1 \pmod 8 = 7 \pmod 8$.

So the final five digits of $5^{(7^{(5^7)})}$ are the final five digits of 5^7,

which are 78125.

Question 29

$0 \leq x \leq 430$, $0 \leq y \leq 2020$.

If $43x = 1 \pmod{431}$, find x.

If $201y = 1 \pmod{2011}$, find y.

What is $x + y$?

Potentially useful facts: if a and b are integers then $a = b \pmod{n}$

if a and b have the same remainder when divided by n.

$a = b \pmod{n} \Rightarrow ka = kb \pmod{n}$, where k is any integer.

To get your final answer, multiply this by 6.7 and take the integer part.

(16227)

Correct answer releases

$m_{11} = 1$, $m_{12} = -1$, $m_{21} = 4$, $m_{22} = 1$

⇩

Solution to Question 29

We need 43x + 431k = 1, so 430x + 4310k = 10

So –x = 10 (mod 431), and x = 421.

We also need 201y + 2011j = 1, so 2010y + 20110j = 10,

So –y = 10 (mod 2011), and y = 2001.

Thus x + y = 2422.

Question 30

Roger defines a function *R* that acts on a pair of values (x, y),

turning them into $R(x,y) = \left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y, \frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)$.

So for example, $R(0,0) = (0,0)$ and $R(1,0) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

He defines $R^2(x,y)$ as the result of applying the function *R* twice,

i.e. $R^2(x,y) = R(R(x,y))$,

$R^3(x,y)$ as the result of applying *R* three times, and so on.

He defines the pair of values (a, b) as equal to $R^{2021}(1,0)$.

Work out and submit the value of $|a + b|$ to three significant figures.

To get your final answer, multiply this by 98765 and take the integer part.

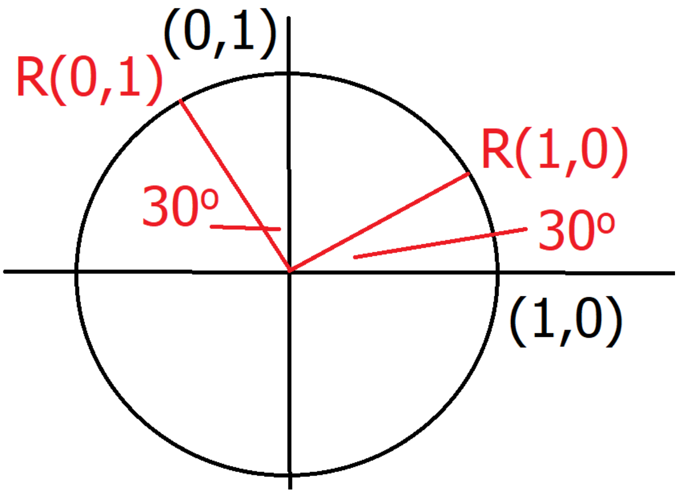
(36147)

Correct answer releases

Convert numbers γ and δ to letters using the conversion table

Solution to Question 30

Where does R send (1, 0) and (0, 1)?



R must be a rotation of 30º about the origin anticlockwise.

R can be represented by a 2 by 2 matrix, as below.

$$\begin{pmatrix} \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 30^o & -\sin 30^o \\ \sin 30^o & \cos 30^o \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Fact: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the matrix representing

a rotation about the origin anticlockwise through θ.

Thus R^{12} = I, where I is the identity transformation

that leaves everything where it is.

2021 = 12 × 168 + 5, so R^{2021} has the same effect as R^5,

which is a rotation through 150º anticlockwise about O.

$$\begin{pmatrix} \cos 150^o & -\sin 150^o \\ \sin 150^o & \cos 150^o \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.866025... \\ 0.5 \end{pmatrix} \Rightarrow |a+b| = 0.366(3s.f.)$$

Question 31

Sanvitha defines a function S that reverses the action of Roger's function R,

so that $S(R(x,y)) = (x,y)$.

For example, $S(0,0) = (0,0)$ and $S\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right) = (1,0)$.

We can say that S 'undoes' R, and that S is the inverse of R.

Potentially useful fact: if R is represented by the matrix M,

then S is represented by the inverse of M = M^{-1} .

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

ad – bc is called the determinant of M.

How do we multiply two matrices?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$$

She defines the pair of values (c,d) as equal to $S(0,2)$.

Work out and submit the value of |c + d| to three significant figures.

To get your final answer, multiply this by 23456 and take the integer part.

(64034)

Correct answer releases

$$\gamma = m_{11}\alpha + m_{12}\beta \pmod{26}$$
$$\delta = m_{21}\alpha + m_{22}\beta \pmod{26}$$

Solution to Question 31

Reversing a rotation through 30^o anticlockwise about O

means just making the direction clockwise instead of anticlockwise.

Important facts;

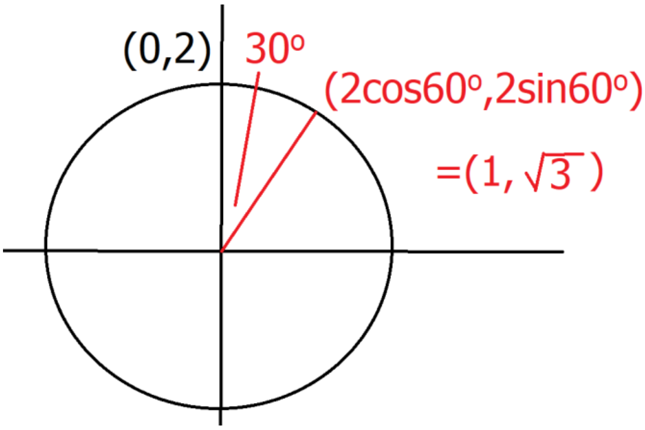
cos(-x) = cos x, sin (-x) = -sin x.

$$\begin{pmatrix} \cos(-30^o) & -\sin(-30^o) \\ \sin(-30^o) & \cos(-30^o) \end{pmatrix} = \begin{pmatrix} \cos 30^o & \sin 30^o \\ -\sin 30^o & \cos 30^o \end{pmatrix}$$

So we need $\begin{pmatrix} \cos 30^o & \sin 30^o \\ -\sin 30^o & \cos 30^o \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

Thus c + d = 2.73(3s.f.)

Or alternatively,



Question 32

Potentially useful fact: $\frac{1}{p} = q(\text{mod n}) \;\Leftrightarrow\; 1 = pq(\text{mod n})$

Find all pairs of natural numbers (*x*, *y*) where 0 < x < 26, 0 < y < 26

for which *xy* = 1 (mod 26).

Take (y, x) to be the same pair as (x, y),

and order each pair so that in (x, y), x ≤ y.

Order the pairs in ascending order of the first number in each pair.

Concatenate all the numbers into one long number.

The first pair is (1, 1) so the long number begins '11'.

To get your final answer, form a five digit number from the

third, seventh, tenth, thirteenth and twentieth digits here.

(31515)

Correct answer releases

Convert letters to numbers α and β
using the conversion table

Solution to Question 32

Excel gives us a quick answer.
Drawing up a table can be a good way to tackle mod problems.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	0	2	4	6	8	10	12	14	16	18	20	22	24	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	1	4	7	10	13	16	19	22	25	2	5	8	11	14	17	20	23
4	0	4	8	12	16	20	24	2	6	10	14	18	22	0	4	8	12	16	20	24	2	6	10	14	18	22
5	0	5	10	15	20	25	4	9	14	19	24	3	8	13	18	23	2	7	12	17	22	1	6	11	16	21
6	0	6	12	18	24	4	10	16	22	2	8	14	20	0	6	12	18	24	4	10	16	22	2	8	14	20
7	0	7	14	21	2	9	16	23	4	11	18	25	6	13	20	1	8	15	22	3	10	17	24	5	12	19
8	0	8	16	24	6	14	22	4	12	20	2	10	18	0	8	16	24	6	14	22	4	12	20	2	10	18
9	0	9	18	1	10	19	2	11	20	3	12	21	4	13	22	5	14	23	6	15	24	7	16	25	8	17
10	0	10	20	4	14	24	8	18	2	12	22	6	16	0	10	20	4	14	24	8	18	2	12	22	6	16
11	0	11	22	7	18	3	14	25	10	21	6	17	2	13	24	9	20	5	16	1	12	23	8	19	4	15
12	0	12	24	10	22	8	20	6	18	4	16	2	14	0	12	24	10	22	8	20	6	18	4	16	2	14
13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13
14	0	14	2	16	4	18	6	20	8	22	10	24	12	0	14	2	16	4	18	6	20	8	22	10	24	12
15	0	15	4	19	8	23	12	1	16	5	20	9	24	13	2	17	6	21	10	25	14	3	18	7	22	11
16	0	16	6	22	12	2	18	8	24	14	4	20	10	0	16	6	22	12	2	18	8	24	14	4	20	10
17	0	17	8	25	16	7	24	15	6	23	14	5	22	13	4	21	12	3	20	11	2	19	10	1	18	9
18	0	18	10	2	20	12	4	22	14	6	24	16	8	0	18	10	2	20	12	4	22	14	6	24	16	8
19	0	19	12	5	24	17	10	3	22	15	8	1	20	13	6	25	18	11	4	23	16	9	2	21	14	7
20	0	20	14	8	2	22	16	10	4	24	18	12	6	0	20	14	8	2	22	16	10	4	24	18	12	6
21	0	21	16	11	6	1	22	17	12	7	2	23	18	13	8	3	24	19	14	9	4	25	20	15	10	5
22	0	22	18	14	10	6	2	24	20	16	12	8	4	0	22	18	14	10	6	2	24	20	16	12	8	4
23	0	23	20	17	14	11	8	5	2	25	22	19	16	13	10	7	4	1	24	21	18	15	12	9	6	3
24	0	24	22	20	18	16	14	12	10	8	6	4	2	0	24	22	20	18	16	14	12	10	8	6	4	2
25	0	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

We can see here the solutions are

(1, 1), (3, 9), (5, 21), (7, 15), (11, 19), (17, 23), (25, 25)

So this gives us

1139521715111917232525

The third, seventh, tenth, thirteenth and twentieth digits here give us 31515 as our final answer.

Or else there is a theorem from number theory; given integers u and v,

then $au + bv = 1$ has a solution for some integers a and b

if and only if the highest common factor of u and v is 1.

Using this, we seek solutions to $xy + 26k = 1$, so x and y cannot have any common factors with 26.

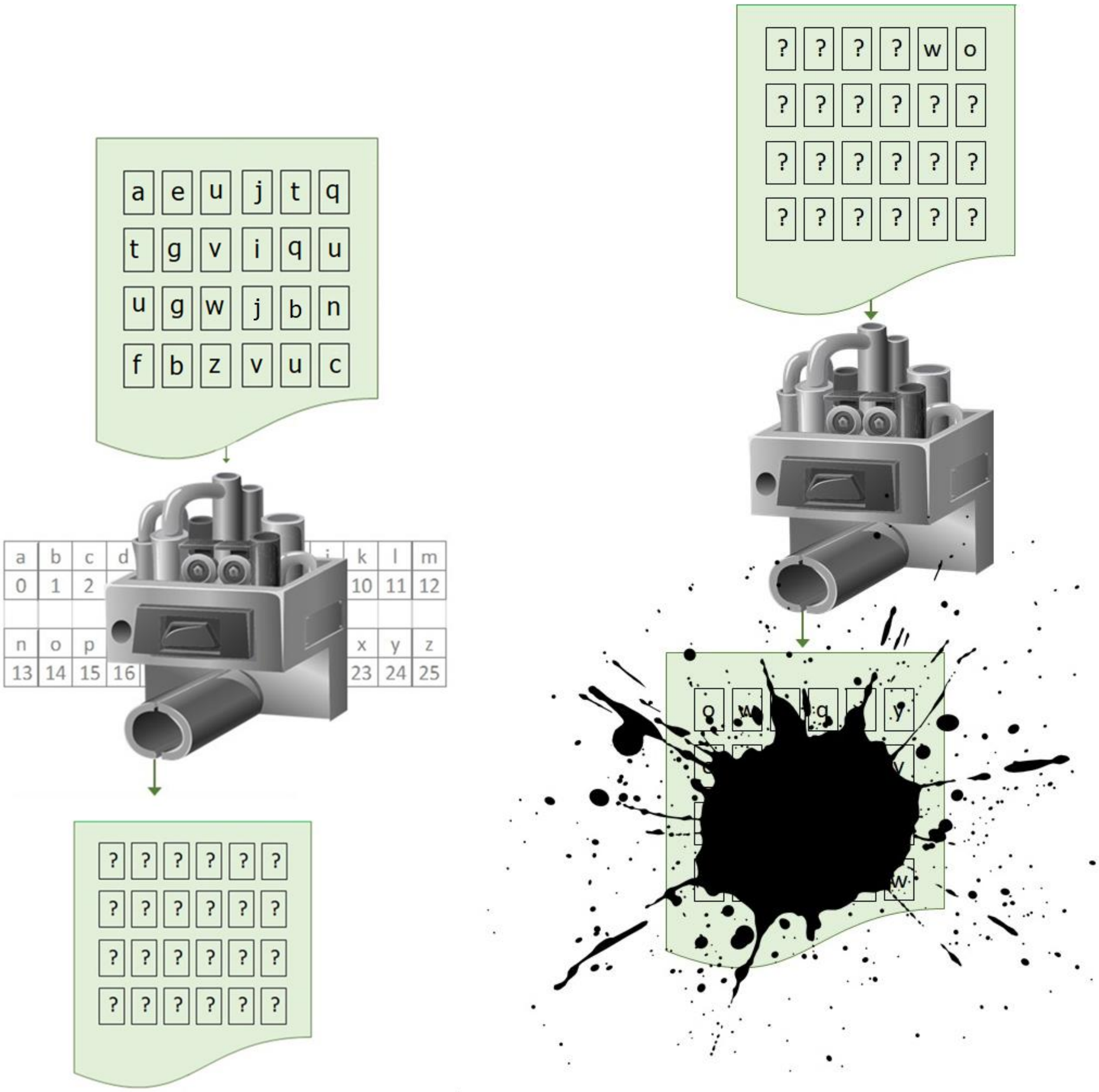
(x and y both have to be odd, and neither of them can be 13)

You can see from our answers above

that all the odd numbers apart from 13 appear as x or y,

and no even numbers appear.

Final Clue for Stage Two

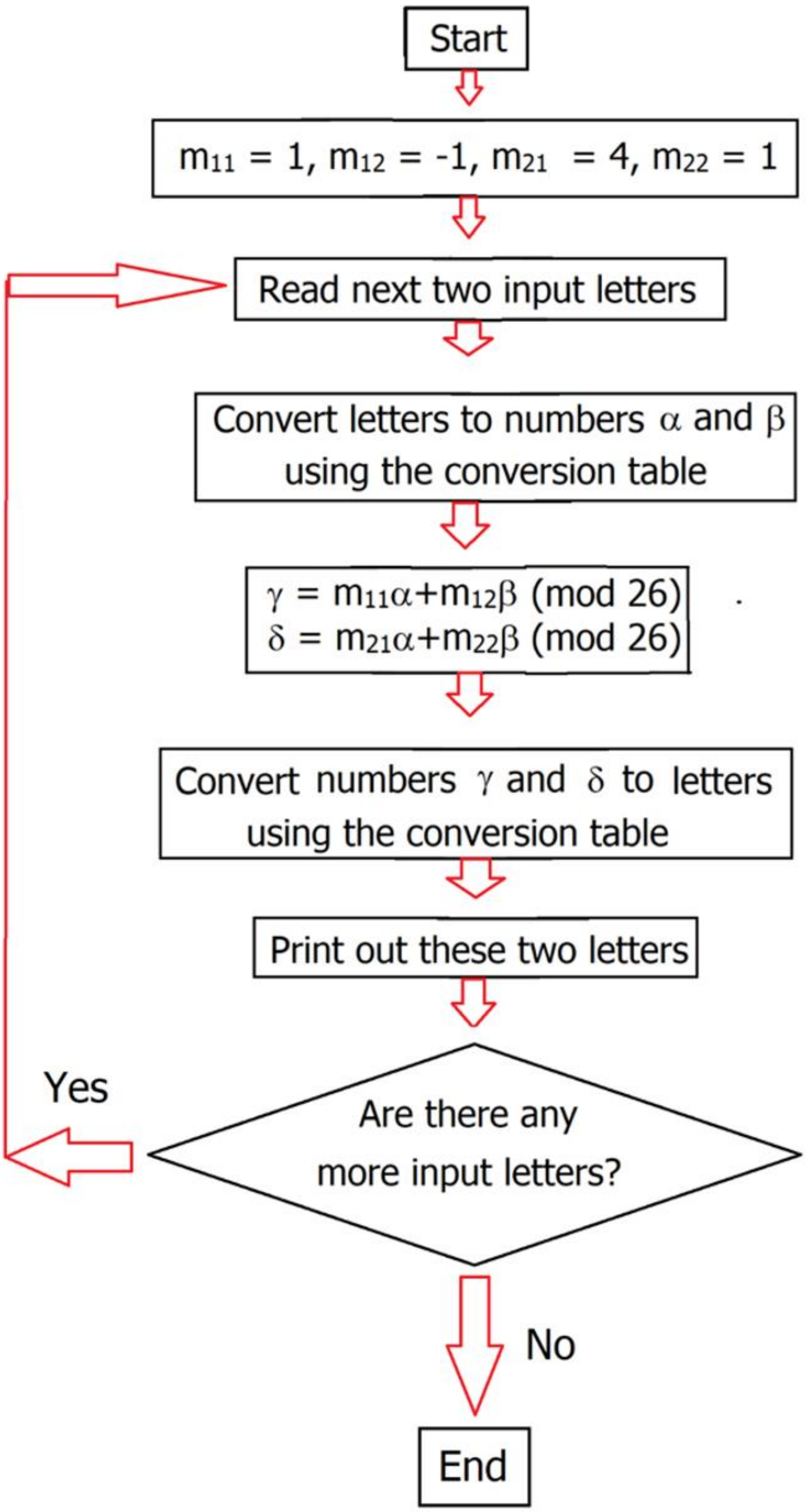


Now enter your sixteen-character string to unlock Stage 3.

Solution to Stage 2

The eight questions release eight parts of a coding machine.

The machine can be assembled as follows;



Starting with the left-hand picture, decoding the given input should give you

welldonenowgointoreverse

The ink splash indicates the 24-character string of letters

given by the clues to questions 1 to 24

is the output to some as yet unidentified input.

We have to find the input,

which involves reversing the machine you have created.

There are at least three methods for reversing the machine.

Firstly, you can use Excel

to create a table to move pairs of letters forwards through the machine.

B2 =MOD(B$1-$A2,26)+0.01*MOD(4*B$1+$A2,26)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA
1		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	0	0.00	1.04	2.08	3.12	4.16	5.20	6.24	7.02	8.06	9.10	10.14	11.18	12.22	13.00	14.04	15.08	16.12	17.16	18.20	19.24	20.02	21.06	22.10	23.14	24.18	25.22
3	1	25.01	0.05	1.09	2.13	3.17	4.21	5.25	6.03	7.07	8.11	9.15	10.19	11.23	12.01	13.05	14.09	15.13	16.17	17.21	18.25	19.03	20.07	21.11	22.15	23.19	24.23
4	2	24.02	25.06	0.10	1.14	2.18	3.22	4.00	5.04	6.08	7.12	8.16	9.20	10.24	11.02	12.06	13.10	14.14	15.18	16.22	17.00	18.04	19.08	20.12	21.16	22.20	23.24
5	3	23.03	24.07	25.11	0.15	1.19	2.23	3.01	4.05	5.09	6.13	7.17	8.21	9.25	10.03	11.07	12.11	13.15	14.19	15.23	16.01	17.05	18.09	19.13	20.17	21.21	22.25
6	4	22.04	23.08	24.12	25.16	0.20	1.24	2.02	3.06	4.10	5.14	6.18	7.22	8.00	9.04	10.08	11.12	12.16	13.20	14.24	15.02	16.06	17.10	18.14	19.18	20.22	21.00
7	5	21.05	22.09	23.13	24.17	25.21	0.25	1.03	2.07	3.11	4.15	5.19	6.23	7.01	8.05	9.09	10.13	11.17	12.21	13.25	14.03	15.07	16.11	17.15	18.19	19.23	20.01
8	6	20.06	21.10	22.14	23.18	24.22	25.00	0.04	1.08	2.12	3.16	4.20	5.24	6.02	7.06	8.10	9.14	10.18	11.22	12.00	13.04	14.08	15.12	16.16	17.20	18.24	19.02
9	7	19.07	20.11	21.15	22.19	23.23	24.01	25.05	0.09	1.13	2.17	3.21	4.25	5.03	6.07	7.11	8.15	9.19	10.23	11.01	12.05	13.09	14.13	15.17	16.21	17.25	18.03
10	8	18.08	19.12	20.16	21.20	22.24	23.02	24.06	25.10	0.14	1.18	2.22	3.00	4.04	5.08	6.12	7.16	8.20	9.24	10.02	11.06	12.10	13.14	14.18	15.22	16.00	17.04
11	9	17.09	18.13	19.17	20.21	21.25	22.03	23.07	24.11	25.15	0.19	1.23	2.01	3.05	4.09	5.13	6.17	7.21	8.25	9.03	10.07	11.11	12.15	13.19	14.23	15.01	16.05
12	10	16.10	17.14	18.18	19.22	20.00	21.04	22.08	23.12	24.16	25.20	0.24	1.02	2.06	3.10	4.14	5.18	6.22	7.00	8.04	9.08	10.12	11.16	12.20	13.24	14.02	15.06
13	11	15.11	16.15	17.19	18.23	19.01	20.05	21.09	22.13	23.17	24.21	25.25	0.03	1.07	2.11	3.15	4.19	5.23	6.01	7.05	8.09	9.13	10.17	11.21	12.25	13.03	14.07
14	12	14.12	15.16	16.20	17.24	18.02	19.06	20.10	21.14	22.18	23.22	24.00	25.04	0.08	1.12	2.16	3.20	4.24	5.02	6.06	7.10	8.14	9.18	10.22	11.00	12.04	13.08
15	13	13.13	14.17	15.21	16.25	17.03	18.07	19.11	20.15	21.19	22.23	23.01	24.05	25.09	0.13	1.17	2.21	3.25	4.03	5.07	6.11	7.15	8.19	9.23	10.01	11.05	12.09
16	14	12.14	13.18	14.22	15.00	16.04	17.08	18.12	19.16	20.20	21.24	22.02	23.06	24.10	25.14	0.18	1.22	2.00	3.04	4.08	5.12	6.16	7.20	8.24	9.02	10.06	11.10
17	15	11.15	12.19	13.23	14.01	15.05	16.09	17.13	18.17	19.21	20.25	21.03	22.07	23.11	24.15	25.19	0.23	1.01	2.05	3.09	4.13	5.17	6.21	7.25	8.03	9.07	10.11
18	16	10.16	11.20	12.24	13.02	14.06	15.10	16.14	17.18	18.22	19.00	20.04	21.08	22.12	23.16	24.20	25.24	0.02	1.06	2.10	3.14	4.18	5.22	6.00	7.04	8.08	9.12
19	17	9.17	10.21	11.25	12.03	13.07	14.11	15.15	16.19	17.23	18.01	19.05	20.09	21.13	22.17	23.21	24.25	25.03	0.07	1.11	2.15	3.19	4.23	5.01	6.05	7.09	8.13
20	18	8.18	9.22	10.00	11.04	12.08	13.12	14.16	15.20	16.24	17.02	18.06	19.10	20.14	21.18	22.22	23.00	24.04	25.08	0.12	1.16	2.20	3.24	4.02	5.06	6.10	7.14
21	19	7.19	8.23	9.01	10.05	11.09	12.13	13.17	14.21	15.25	16.03	17.07	18.11	19.15	20.19	21.23	22.01	23.05	24.09	25.13	0.17	1.21	2.25	3.03	4.07	5.11	6.15
22	20	6.20	7.24	8.02	9.06	10.10	11.14	12.18	13.22	14.00	15.04	16.08	17.12	18.16	19.20	20.24	21.02	22.06	23.10	24.14	25.18	0.22	1.00	2.04	3.08	4.12	5.16
23	21	5.21	6.25	7.03	8.07	9.11	10.15	11.19	12.23	13.01	14.05	15.09	16.13	17.17	18.21	19.25	20.03	21.07	22.11	23.15	24.19	25.23	0.01	1.05	2.09	3.13	4.17
24	22	4.22	5.00	6.04	7.08	8.12	9.16	10.20	11.24	12.02	13.06	14.10	15.14	16.18	17.22	18.00	19.04	20.08	21.12	22.16	23.20	24.24	25.02	0.06	1.10	2.14	3.18
25	23	3.23	4.01	5.05	6.09	7.13	8.17	9.21	10.25	11.03	12.07	13.11	14.15	15.19	16.23	17.01	18.05	19.09	20.13	21.17	22.21	23.25	24.03	25.07	0.11	1.15	2.19
26	24	2.24	3.02	4.06	5.10	6.14	7.18	8.22	9.00	10.04	11.08	12.12	13.16	14.20	15.24	16.02	17.06	18.10	19.14	20.18	21.22	22.00	23.04	24.08	25.12	0.16	1.20
27	25	1.25	2.03	3.07	4.11	5.15	6.19	7.23	8.01	9.05	10.09	11.13	12.17	13.21	14.25	15.03	16.07	17.11	18.15	19.19	20.23	21.01	22.05	23.09	24.13	25.17	0.21

The horizontal line in yellow is the first letter of the pair,

while the vertical yellow line is the second letter of the pair.

Using Excel's mod function,

we can find the output pair of letters for any input pair.

(Note the use of 0.01 in each cell to give the output letters side by side).

All the columns contain the numbers from 0 to 25 twice,

once before the dot and once after the dot.

The process is easily reversed; just find the pair of letters you are looking for

(once converted into numbers) in the grid,

and then read off the yellow numbers.

If you would rather convert letters to letters

you could use VLOOKUP() or XLOOKUP() to refine the table.

Secondly, you can solve the mod 26 simultaneous equations

that are given in the machine.

We have $\gamma = \alpha - \beta (\text{mod } 26),\ \delta = 4\alpha + \beta (\text{mod } 26)$.

Adding, $\gamma + \delta = 5\alpha (\text{mod } 26)$.

Note that 5×21 - 4×26 = 1.

(Question 32 is a help with seeing this).

So multiplying this equation by 21,

$21\gamma + 21\delta = 105\alpha = \alpha (\text{mod } 26)$.

$\beta = \alpha - \gamma\ (\text{mod } 26) = 20\gamma + 21\delta\ (\text{mod } 26)$

So now we have α and β in terms of γ and δ,

and we can use these equations to work backwards.

Just input the letters γ and δ, once converted into numbers,

into the equations, and you find α and β (mod 26).

Thirdly, we can think in terms of matrices.

Forwards through the machine, we are using the matrix

$$M = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}.$$

We need to find the inverse of this (mod 26).

The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(try multiplying these matrices together and see what happens!)

The expression ad - bc is called the ***determinant*** of the matrix.

So how do we find $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$(mod 26)?

What is $\frac{1}{ad-bc}$(mod 26)?

We need to find x, where x(ad - bc) = 1 (mod 26).

Now for our matrix, ad - bc = 5.

So can we solve 5x = 1 (mod 26)?

Note that 5×21 - 4×26 = 1.

(Question 32 is a help with seeing this).

So x is 21 for our matrix,

and we can replace $\frac{1}{ad-bc}$ with 21 in $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$(mod 26).

So we need to find $21\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$(mod 26)

$$= \begin{pmatrix} 21 & 21 \\ -84 & 21 \end{pmatrix} (\text{mod } 26) = \begin{pmatrix} 21 & 21 \\ 20 & 21 \end{pmatrix} (\text{mod } 26)$$

Using this matrix, we can now map an output pair of letters

to an input pair of letters.

However you tackle this problem,

you should find that the mystery input in the right-hand machine picture is

codewordritangletriangle

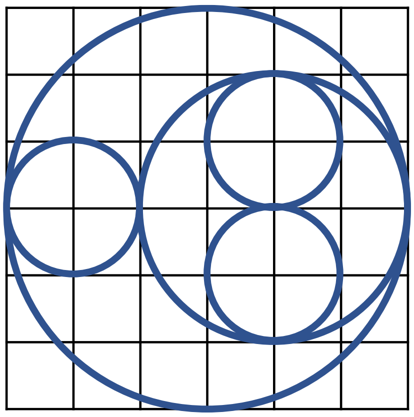
Substituting this codeword into the webpage should release Stage 3 to you.

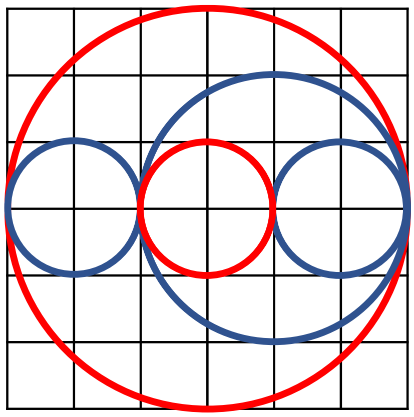
Ritangle 2021, Stage 3

Consider the following problem. You have a 6×6 grid as shown, with area 36 square units. Your task is to place a number of circles on the grid (as many as you wish), subject to the following rules:

1. Each circle must be centred at one of the grid points. No two circles may be centred on the same grid point.
2. The radius of each circle must be a whole number of units.
3. Circles may touch the boundary of the grid but may not cross it.
4. Circles may touch one another (at a tangent) but their perimeters may not cross. It is allowed for one circle to be entirely inside another.
5. You must maximise the total area enclosed by all your circles.

The diagrams below show two candidate solutions, each giving a total area that is 16 times the area of the unit circle. However, the second one is not legitimate because the centres of the two red circles coincide. The first diagram is a legitimate solution and in fact gives the maximum total area.





The final stage of Ritangle 2021 requires you to solve a similar problem but in three dimensions.

You have a cuboidal grid that measures u units × v units × w units where u, v and w are positive integers. Assume that the box occupies the space from (0, 0, 0) to (u, v, w).

You have to place a number of spheres in the grid, subject to the following rules:

1. Each sphere must be centred at one of the grid points. No two spheres may be centred on the same grid point.
2. The radius of each sphere must be a whole number of units.
3. Spheres may touch the boundary of the grid but may not cross it.
4. Spheres may touch one another but their boundaries may not cross. It is allowed for one sphere to be entirely inside another.
5. You may choose the values of *u*, *v* and *w*, subject to the constraint $u + v + w \leq 51$.
6. You must maximise the total volume enclosed by all your spheres.

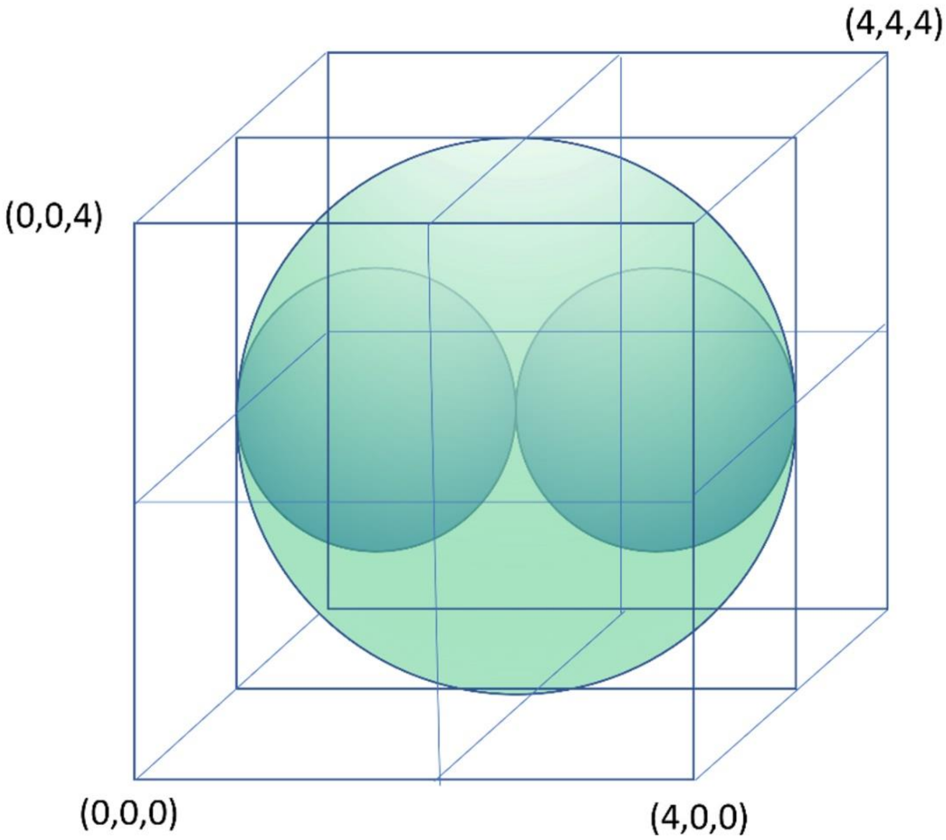
Express the total volume *V* as a multiple of the volume of a sphere with radius 1 unit (i.e. calculate the volume in cubic units and divide by $\frac{4}{3}\pi$). Submit your answer in the following form;

u, *v*, *w*
V
x_1, y_1, z_1, r_1
x_2, y_2, z_2, r_2
x_3, y_3, z_3, r_3
…

where each set x_i, y_i, z_i, r_i gives the centre and the radius of sphere *i*, and there is one line for each sphere in your solution. You should include commas but not spaces.

For example, the solution pictured below for a 4×4×4 grid (for clarity, only the even-numbered grid lines are shown in the diagram) would be submitted as follows;

4,4,4
10
1,2,2,1
3,2,2,1
2,2,2,2


(4,4,4)
(0,0,4)
(0,0,0)
(4,0,0)

Solution to Stage 3

We explored Stage 3 using Excel, which helped to direct our thoughts. We also used Excel to score and validate submitted solutions.

Our Excel file has:

1. a worksheet ***scoringLive*** that takes the submitted results, checks that they're valid (overlapping spheres generate a red cell) and calculates a score (currently showing at top left)

2. a worksheet ***scoreArchive*** that provides an archive of submitted results

3. three explanatory worksheets, ***Spheres12_easy***, ***Spheres16_better*** and ***SpheresUVW_vgood*** a journey through

- the 12x12x12 case (which is easy, but shows the diagramming conventions used on later pages)
- the 16x16x16 case (harder, but manageable)
- the general (u, v, w) case, which offers a wider solution space to explore and demands a bit of extra thinking.

Pleasingly, because of the constraints on integer radii and all spheres being centred on grid points, the largest possible box (17x17x17) doesn't give the largest enclosed volume of the spheres.

Programming is not an essential requirement with this problem: Excel formulae should cover everything.

You will find the Excel file on the Ritangle 2021 webpage.

So that's it!

Ritangle 2021 comes to an end;

if you made it this far, well done,

if you paused for breath along the way,

we hope you learnt something from the maths you did.

Hoping to see you next year!

Jonny, Roger, Bernard and all the Ritangle team