Ritangle - The Integral A Level Maths Competition 2017

Questions and Answers

The Preliminary Questions

A. The 7-digit number 3211000 is called *self-descriptive* since it contains three 0s, two 1s, one 2, one 3, zero 4s, zero 5s, zero 6s and zero 7s. Find the two smallest self-descriptive numbers and add them together.

Note; a self-descriptive number does not have to have seven digits.

Answer to A; It is easy to check that there are no one or two-digit self-descriptive numbers. Suppose $n_0 n_1 n_2$ is a three-digit self-descriptive number, and so $n_0 + n_1 + n_2 = 3$ and $n_1 + 2n_2 = 3$. This tells us that $n_0 = n_2$, and it is easy to check this can never be. What about $n = 4$? Here we need $n_0 + n_1 + n_2 + n_3 = 4$ and $n_1 + 2n_2 + 3n_3 = 4$. This tells us that $n_0 = n_2 + 2n_3$. Checking reveals that $n_0 = 1, n_2 = 1, n_3 = 0$ and $n_0 = 2, n_2 = 2, n_3 = 0$ both work here, and that these are the only possibilities. Thus there are two four-digit self-descriptive numbers, 2020 and 1210, which add to **3230**.

B. You are given nine rods of lengths $6, 7, 8, 10, 15, 17, 24, 25$ and 26. You pick three at random, and p is the probability that you can form a triangle with your rods.

The choice $(6, 7, 26)$ is a fail, and so is $(7, 10, 17)$.

In addition, q is the probability that your three rods make a right-angled triangle. What is q/p? Multiply your answer by 1000 and round to the nearest integer.

Answer to B; The total number of different choices for our three rods is ${}^9C_3 = 84$.

We can make a spreadsheet that gives the longest side of a right-angled triangle from the other two sides.

	6	7	8	10	15	17	24	25	26
6	8.485281	9.219544	10	11.6619	16.15549	18.02776	24.73863	25.70992	26.68333
7	9.219544	9.899495	10.63015	12.20656	16.55295	18.38478	25	25.96151	26.92582
8	10	10.63015	11.31371	12.80625	17	18.78829	25.29822	26.24881	27.20294
10	11.6619	12.20656	12.80625	14.14214	18.02776	19.72308	26	26.92582	27.85678
15	16.15549	16.55295	17	18.02776	21.2132	22.67157	28.30194	29.15476	30.01666
17	18.02776	18.38478	18.78829	19.72308	22.67157	24.04163	29.41088	30.23243	31.06445
24	24.73863	25	25.29822	26	28.30194	29.41088	33.94113	34.65545	35.38361
25	25.70992	25.96151	26.24881	26.92582	29.15476	30.23243	34.65545	35.35534	36.06938
26	26.68333	26.92582	27.20294	27.85678	30.01666	31.06445	35.38361	36.06938	36.76955

We can see that $(6, 8, 10), (8, 15, 17), (7, 24, 25)$, and $(10, 24, 26)$ are the only rod choices that give us right-angled triangles, and so $q = \dfrac{4}{84}$.

To find the number of choices of three rods that can make a viable triangle requires counting. The key thing is that the two shortest sides when added together must exceed the third side. If the two shortest sides are 6 and 7, then the longest side could only be 8 or 10. If the two shortest sides are 6 and 8, then the longest side could only 10. If we carry on in this way, we can count 8 triangles with 6 as the shortest side. Considering all sides logically in this way, we get a total of 39.

(6,7,8),(6,7,10),(6,8,10),(6,10,15),(6,15,17)(6,24,25),(6,24,26),(6,25,26),

(7,8,10),(7,10,15),(7,15,17),(7,24,25),(7,24,26),(7,25,26),

(8,10,15),(8,10,17),(8,15,17),(8,17,24),(8,24,25),(8,24,26),(8,25,26),

(10,15,17),(10,15,24),(10,17,24),(10,17,25),(10,17,26),(10,24,25),(10,24,26),(10,25,26),

(15,17,24),(15,17,25),(15,17,26),(15,24,25),(15,24,26),(15,25,26),

(17,24,25),(17,24,26),(17,25,26) and (24,25,26).

Thus $p = \frac{39}{84}$, and $\frac{q}{p} = \frac{4}{39} = 0.1025...$, and our answer is **103**.

C. Two competing shops have a suit for sale, and both are asking for the same price. Both shops have a sale; the first shop drops the price of the suit by £18, the second drops it by 18%. The following week, the first shop drops the price of the suit by a further 21%, while the second shop takes off a further £21. After this second round of deductions, the two shops are again offering the suit at the same price. What was the original price of the suit in pounds?

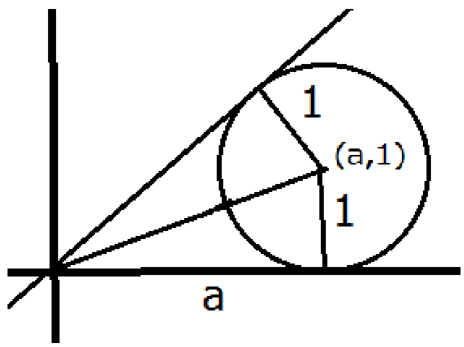
Answer to C; let the starting price be £a. Then $(a-18)\dfrac{79}{100} = \left(a \times \dfrac{82}{100}\right) - 18$. Solving this equation for a gives $a = 226$, and so our answer is **226**.

D. A triangle ABC has a perimeter of P cm and an area of Q cm^2, where $P = 2Q$. Triangle DEF is similar to ABC. The combined perimeter of the two triangles in cm is equal numerically to their combined area in cm^2. If DEF has an area k times larger than ABC, what is k? Multiply your answer by 100 and round to the nearest integer.

Answer to D: The triangle ABC has perimeter $2A$ and area A, while triangle DEF has perimeter $2jA$ and area j^2A for some scale factor j, where j is positive. Thus we have $2A + 2jA = A + j^2A$, and so $j^2 - 2j - 1 = 0$. This gives us that $j = 1 + \sqrt{2}$ (the other value for j is negative). The value $k = j^2 = 3 + 2\sqrt{2} = 5.8284...$ and so our answer is **583**.

E. A circle of radius 1 rolls along the positive x-axis towards the origin until it is stopped by the line $y = x$. What is the x-coordinate of its centre now? Multiply your answer by 1000 and round to the nearest integer.

Answer to E:



From the diagram we can see that we want to find a. The equation of the circle is $(x - a)^2 + (y - 1)^2 = 1$. Now the line $y = x$ is a tangent to this, so solving this equation together with $y = x$ must give equal roots. So solving simultaneously we have $(y - a)^2 + (y - 1)^2 = 1$, which multiplies out to $2y^2 + 9 - 2a - 2)y = a^2 = 0$. Since the two roots here are equal, the discriminant must be zero, so $(-2a - 2)^2 = 4 \times 2 \times a^2$. This simplifies to $a^2 - 2a - 1 = 0$, and so $a = 1 + \sqrt{2} \approx 2.4142$, and so our answer is **2414**.

Alternatively you can say that the the angle that $y = x$ makes with the x-axis at the origin is $45°$, and so halving that gives $a = \cot(22.5°) = 2.414...$

The Main Puzzle Questions with Answers

1.

1 2 3
4 5 6

In the diagram above, you are permitted to journey horizontally and vertically from any start point.

You are also allowed to retrace your steps.

Thus possible journeys include, 1, 12, 214, 123654, 12541, 12321, 33333.

The journeys 233 and 126 are impossible.

The diagram below contains the journeys for the first thirteen numbers from a simple sequence where all the terms are different, increasing and positive.

What is the fourteenth number?

1 4 6 1
3 3 9 2
5 1 2 5
1 9 7 1
0 0 2 8

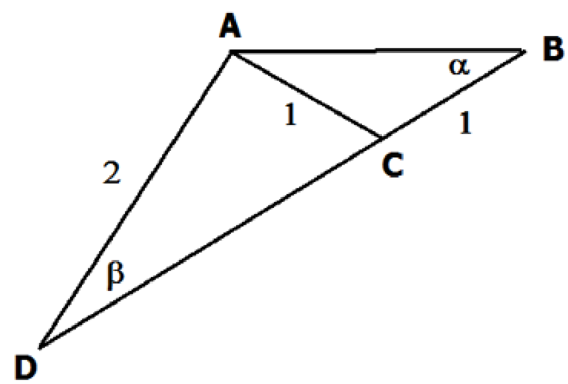
Answer to 1; The numbers $1, 2, 3, 4...$ can't be found since there is no 11. The numbers $1^2, 2^2, 3^2, 4^2...$ can't be found since there is no 36. But $1^3, 2^3, 3^3, 4^3...13^3$ can all be spelled out (a journey is allowed to go back on itself, so 1000 is possible). Thus the fourteenth in the sequence is $14^3 = \mathbf{2744}$.

2. What is u_{101} in this sequence?

$$u_1 = \textbf{thousand}, u_2 = \textbf{million}, u_3 = \textbf{billion}, ...$$

Answer to 2; this sequence is formed by adding on three 0s each time. The last of these powers of 10 to be named is 1 followed by 303 zeroes which is a **centillion**.

3.



A triangle ABC is isosceles, with $AC = BC = 1$. The point D lies on BC produced so that $AD = 2$. Angle $ABC = \alpha$, while angle $ADC = \beta$, and $\alpha + \beta = \frac{\pi}{6}$. What is CD? Multiply your answer by 100 and round to the nearest integer.

Note: we think the answer is very hard to find exactly! We expect you will need graphing software here to help you find the approximate solution to the required accuracy.

Answer to 3: Let $\sin\alpha = s, \cos\alpha = c, CD = z$. Using the sin rule in the triangle ABD, we have $\frac{\sin(\pi - \alpha - \beta)}{z+1} = \frac{s}{2}$.

Since $\sin(\pi - \theta) = \sin\theta$, and $\alpha + \beta = \frac{\pi}{6}$, we have that $\sin(\pi - \alpha - \beta) = 0.5$. Thus $s = \frac{1}{1+z}$.

Using the cosine rule in the triangle ABD, we have $4 = 4c^2 + (1+z)^2 - 4c^2(z+1)$.

Substituting $c^2 = 1 - s^2, s = \frac{1}{1+z}$ and rearranging, we arrive at $(z+1)^4 - 4(z+1)^3 + 4(z+1) - 4 = 0$.

Drawing the graph $y = (x+1)^4 - 4(x+1)^3 + 4(x+1) - 4$ and zooming in on the only possible root gives you $z = 2.795...$

Thus our answer is **280**.

Alternatively you can draw a graph earlier in your working, like this.

Using the sin rule again in ABD, you find $\frac{s}{2} = \frac{2c}{\sin\beta} = \frac{2c}{\sin\left(\frac{\pi}{6} - \alpha\right)}$.

This gives you that $2\sin\left(\frac{\pi}{6} - \alpha\right) = 2\sin\alpha\cos\alpha$.

Now drawing the curve $y = 2\sin\left(\frac{\pi}{6} - x\right) - 2\sin x\cos x$ you can see there is only one relevant zero, at $x = 0.26662...$

So $z = \frac{1}{\sin 0.26662...} - 1 = 2.795...$

4. Two ants walk together along the x-axis from the origin to 1. At 1 they part company; the first ant goes north a distance 0.9, then west $(0.9)^2$, south $(0.9)^3$, east $(0.9)^4$, north $(0.9)^5$ ... while the second ant travels south 0.8, west $(0.8)^2$, north $(0.8)^3$, east $(0.8)^4$, south $(0.8)^5$ .... The first ant ends up at A, and the second ant at B. If m is the gradient of AB, what is $|m|$? Multiply your answer by 100.

Answer to 4: The x-coordinate of A is $1 - 0.9^2 + 0.9^4 + ...$, while the y-coordinate is $0.9 - 0.9^3 + 0.9^5 - ....$ These are both infinite geometric sequences with common ratio -0.9^2, and using $S_\infty = \dfrac{a}{1-r}$, A becomes $\left(\dfrac{100}{181}, \dfrac{90}{181}\right)$. Similarly the point B is $\left(\dfrac{100}{164}, \dfrac{-80}{164}\right)$, and thus the gradient of AB is -17.2, and so $|m| = 17.2$, and our answer is **1720**.

5. Luke has four tiles, each with a different shape, size and colour, and each bearing a different number. The tiles are circular, square, triangular and hexagonal, and they are blue, yellow, red and green in some order. The sizes are tiny, small, large and huge, and the four numbers are $1000, 2000, 3000$ and 4000. You are given these facts;

1. The yellow tile is circular and bears the number 3000.

2. The tiny tile bears either the number 1000 or the number 4000.

3. The red tile is not square.

4. One of the huge tile and the triangular tile is green, while the other bears the number 2000.

5. The tile bearing the number 2000 is either small or large.

6. The small tile's number is 1000 less than the red tile's number.

Now work out the hexagonal tile's number times the blue tile's number as your answer.

You may find this grid helpful.

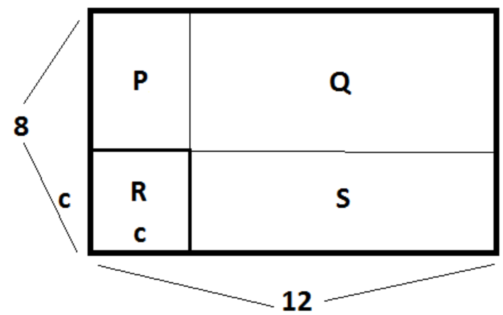
Number	Colour	Shape	Size

Answer to 5; the final table should look like this;

Number	Colour	Shape	Size
1000	Green	Square	Huge
2000	Blue	Triangle	Large
3000	Yellow	Circle	Small
4000	Red	Hexagon	Tiny

And so our answer is **8 000 000.**

6.



A rectangle has sides 12 and 8. A square side c is drawn in one corner, creating the rectangular areas P, Q, R and S as in the diagram. What is the minimum value that $\frac{Q+R}{P+S}$ can take? Multiply your answer by 100 and round to the nearest integer.

Answer to 6;

$$\frac{Q+R}{P+S} = \frac{(12-c)(8-c)+c^2}{(8-c)c+(12-c)c} = \frac{96-20c+2c^2}{20c-2c^2}$$

$$= -1 + \frac{48}{10c-c^2} = \frac{48}{-(c-5)^2+25} - 1.$$

This is a minimum when $(c-5)^2$ is a minimum (which is when it equals 0), and so the minimum value we seeks is $\frac{48}{25} - 1 = 0.92$.

Thus our answer is **92**.

7. We define $\lfloor x \rfloor$ to be the integer part of x, so $\lfloor 45 \rfloor = 45, \lfloor 56.8 \rfloor = 56$. If $u_n = \lfloor (n+1)^{3/2} \rfloor + 1 - 3n$ for $n \geq 1$, and k is the first positive integer so that u_k is positive, what is k?

Answer to 7;

$$
\begin{aligned}
n &= 1; \lfloor (1+1)^{3/2} \rfloor + 1 - 3 = 0, \\
n &= 2; \lfloor (2+1)^{3/2} \rfloor + 1 - 6 = 0, \\
n &= 3; \lfloor (3+1)^{3/2} \rfloor + 1 - 9 = 0, \\
n &= 4; \lfloor (4+1)^{3/2} \rfloor + 1 - 12 = 0, \\
n &= 5; \lfloor (5+1)^{3/2} \rfloor + 1 - 15 = 0, \\
n &= 6; \lfloor (6+1)^{3/2} \rfloor + 1 - 18 = 1.
\end{aligned}
$$

So $k = 6$.

8. The polynomial ax^3+bx^2+cx+1 gives a remainder of 21 when divided by $x-2$, while the polynomial cx^3+ax^2+bx+1 gives a remainder of 25, and the polynomial bx^3+cx^2+ax+1 gives a remainder of -1. Find $a^{(b^c)}$.

Answer to 8; so we have by the remainder theorem $8a+4b+2c+1 = 21, 8c+4a+2b+1 = 25, 8b+4c+2a+1 = -1$.

The first equation tells us $c = -2(2a+b-5)$, and substituting this into the third equation gives us $a = 3$.

Using the second equation now tells us $b = -2$, and so $c = 2$, so $3^{((-2)^2)} = 81$.

9. You are given that $a = 18530, b = 45986, c = 38114$.

Confirm that $a + b, b + c$ and $c + a$ are all perfect squares.

There is a fourth number d so that $a + d = p^2, b + d = q^2$ and $c + d = r^2$, where d, p, q and r are all positive integers. Find d.

Answer to 9; it is easy to check that $18530+38114 = 238^2, 38114+45986 = 290^2, 45986 + 18530 = 254^2$.

We know that $18530 + d = p^2, 38114 + d = q^2, 45986 + d = r^2$.

Subtracting the last two equations gives us that $r^2 - q^2 = (r+q)(r-q) = 7872$.

Now $q > \sqrt{38114} = 195.2...$, while $r > \sqrt{45986} = 214.4...$, so $r + q \geq 411$, which means $r - q \leq \dfrac{7872}{411} = 19.1...$

Using a spreadsheet now is helpful.

q-r	q+r	r	q	d	p
1	7872	3936.5	3935.5	15450046.25	3933.011092
2	3936	1969	1967	3830975	1962.015545
3	2624	1313.5	1310.5	1679296.25	1303.006619
4	1968	986	982	926210	971.977366
5	1574.4	789.7	784.7	577640.09	772.1205152
6	1312	659	653	388295	637.8283468
7	1124.571	565.7857	558.7857	274127.4745	540.9782569
8	984	496	488	200030	467.5040107
9	874.6667	441.8333	432.8333	149230.6944	409.5860037
10	787.2	398.6	388.6	112895.96	362.5271852
11	715.6364	363.3182	352.3182	86014.10124	323.3328026
12	656	334	322	65570	290
13	605.5385	309.2692	296.2692	49661.4571	261.1349404
14	562.2857	288.1429	274.1429	37040.30612	235.733549
15	524.8	269.9	254.9	26860.01	213.0493135
16	492	254	238	18530	192.50974
17	463.0588	240.0294	223.0294	11628.11851	173.6609297
18	437.3333	227.6667	209.6667	5846.111111	156.1285083
19	414.3158	216.6579	197.6579	954.6433518	139.587404

We can see that the value for d we seek is 65570.

10. A rectangle R_1 has sides j and k. The next rectangle in the sequence R_2 has sides $\frac{j}{2}$ and $2k$, while R_3 has sides $\frac{j}{4}$ and $3k$, and R_4 has sides $\frac{j}{8}$ and $4k$, and so on. What in terms of j and k is the sum of the areas of all the rectangles in the sequence?

Answer to 10; the total area is $jk+jk+\frac{3}{4}jk+\frac{4}{8}jk+\frac{5}{16}jk+...+\frac{n}{2^{n-1}}jk+...$

$$= jk\left[\frac{1}{1}+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+...+\frac{n}{2^{n-1}}+...\right]$$

We can write this as in this diagram;

1 +1/2+1/4+1/8+1/16+....
1/2+1/4+1/8+1/16+....
1/4+1/8+1/16+....
1/8+1/16+....
1/16+....

Each row is a geometric series with common ratio 0.5. Remembering that the sum of an infinite geometric series is $S_\infty = \frac{a}{1-r}$, we can add each row and then add these results together, giving

$$jk\left(\frac{1}{0.5}+\frac{1/2}{0.5}+\frac{1/4}{0.5}+\frac{1/8}{0.5}+\frac{1/16}{0.5}+...\right) = 2jk\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...\right) = 4jk.$$

11. Taken in one order the integers $y < z < a$ are consecutive terms from an arithmetic sequence, and taken in another they are three consecutive terms from a geometric sequence.

What is $y^2 + z^2 + a^2$ in terms of z?

Answer to 11; let's say $y = z - d, a = z + d$, where d is positive $(y, z, a$ must be three terms from an increasing AP).

So if $z - d, z$ and $z + d$ are in geometric progression, there are three possibilities;

1.$(z-d)(z+d) = z^2 \implies z^2 - d^2 = z^2$, and so $d = 0$, which is impossible.

2.$(z-d)z = (z+d)^2 \implies d = -3z$.

3.$(z+d)z = (z-d)^2 \implies d = 3z$.

We need to find $(z-d)^2 + z^2 + (z+d)^2$, which is $3z^2 + 2d^2 = 21z^2$ in both cases above (the three terms are $-2z < z < 4z$ or $4z < z < -2z$ depending on whether z is positive or negative).

So the answer is $21z^2$.

12. If a and b are the smallest positive integers so that $5a^7 = 7b^5$, what is ab?

Give your answer to one significant figure.

Answer to 12; we can certainly say 7 divides a, and so some power of 7 divides b. Similarly we can say 5 divides b, and so some power of 5 divides a.

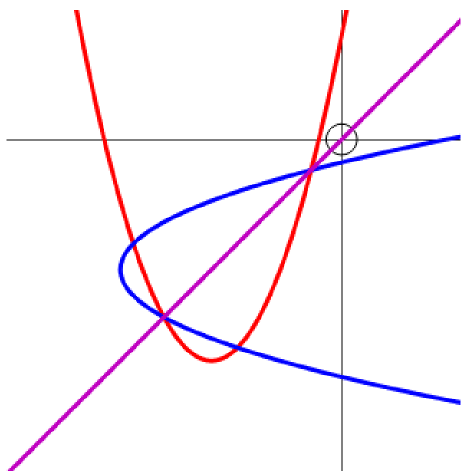
If a and b are to be minimal, then $a = 5^m 7^n, b = 5^t 7^u$. This means $5^{7m+1} 7^{7n} = 5^{5t} 7^{5u+1}$.

So we need the smallest m, n, t, u so that $7m + 1 = 5t, 7n = 5u + 1$. These values are $m = 2, t = 3, n = 3, u = 4$.

So $ab = 5^3 7^4 \times 5^2 7^3 = 2573571875$, and so our answer is **3 billion**.

13. The two parabolas $y = x^2 + 5x + 2$ and $x = y^2 + 5y + 2$ intersect in four points, where two of them (A and B) are on the line $y = x$. What is the distance AB?

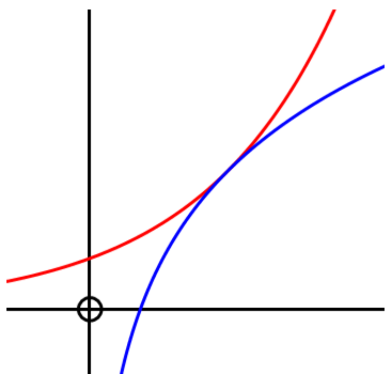
Answer to 13;



Solving the two parabolas simultaneously gives $y^4 + 10y^3 + 34y^2 + 45y + 16 = 0$. This factorises (with the help of the factor theorem or a graphing program) to $(y + 2)(y + 4)(y^2 + 4y + 2) = 0$. We can see from a sketch or calculation that the solution $y = -2, y = -4$ are not the ones that we want, so using the quadratic formula on $y^2 + 4y + 2 = 0$ gives us that $y = -\sqrt{2} - 2$ and $y = \sqrt{2} - 2$ are the solutions we want. So what is the distance between $(-\sqrt{2} - 2, -\sqrt{2} - 2)$ and $(\sqrt{2} - 2, \sqrt{2} - 2)$? Using the distance formula gives us the answer $AB = 4$.

14. For what value of a do the curves $y = a^x$ and $y = \log_a x$ touch? Multiply your answer by 100 and round to the nearest integer. (You will need graphing software to help you with this.)

Answer to 14; If $y = \log_a x$, then $x = a^y$. If we plot the curves $y = a^x$ and $x = a^y$ on the same axes and then vary a, we can see that the two curves touch when $a \approx 1.4447...$, and so our answer is 144.



15. The line L is $2x+3y+q=0$, and the point A is $\left(\frac{1}{5}, \frac{1}{7}\right)$.

Initially A is not on L.

But if we change the x coefficient to $2-a$ then the revised line L goes though A.

If we instead change the y coefficient to $3-b$ then this revised line L goes though A.

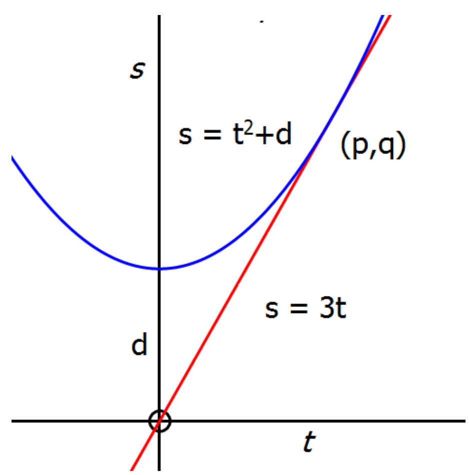
If we instead change the constant term to $q-c$, then this revised line L goes though A.

If $a+b+c=\frac{377}{35}+d$ then what is d in terms of q?

Answer to 15; we must have that A is on $(2-a)x+3y+q=0$, and so $\dfrac{2-a}{5}+\dfrac{3}{7}+q=0$, and so $a=\frac{35q+29}{7}$. Similarly $\dfrac{2}{5}+\dfrac{3-b}{7}+q=0$, and so $b=\frac{35q+29}{5}$, and $\dfrac{2}{5}+\dfrac{3}{7}+(c-q)=0$, and so $c=q-\frac{29}{35}$. Thus $a+b+c=\dfrac{377}{35}+13q$, and so the missing term we seek is $13q$.

16. Fred the policeman sees the man he wants, Roger the burglar, in a car down the straight road ahead. Fred is cycling along at a steady speed of 3m/s. As he passes a lamp-post Roger spots him and starts to drive away from rest with a steady acceleration of $\frac{9}{170}$m/s^2. Fred's front wheel just grazes Roger's back bumper before Roger disappears into the distance. How far, in metres, was Roger's car initially from the lamp-post? Multiply your answer by 100.

Answer to 16; Let's put time $= t = 0$ as Fred passes the lamp-post. Roger's acceleration is $\frac{9}{170}$, so his velocity after t seconds is $0.1t$ (since he starts from rest), and his distance from the lamp-post is $0.05t^2 + d$. Fred's distance from the lamp-post is $3t$ at time t. A distance-time graph looks like this (it includes the fact that Fred only just catches up with the car);



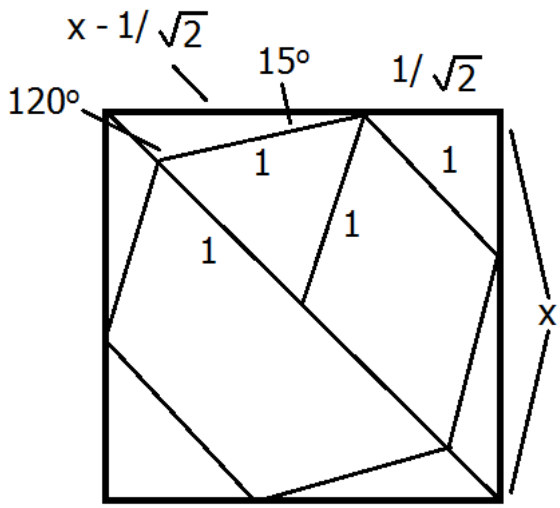
The point (p, q) is on both the line and the curve, so $q = 3p, q = \frac{9}{340}p^2 + d$.

Thus we have $9p^2 - 1020p + 340d = 0$. Since the bike and car only touch, this must have equal roots, and so the discriminant is zero.

So $1020^2 = 4 \times 9 \times 340d$ and $d = 85$m, and so our answer is 8500.

17. A square contains the largest possible regular hexagon that it can. What is $\frac{\text{area of hexagon}}{\text{area of square}}$? Multiply your answer by 1000 and round to the nearest integer.

Answer to 17;

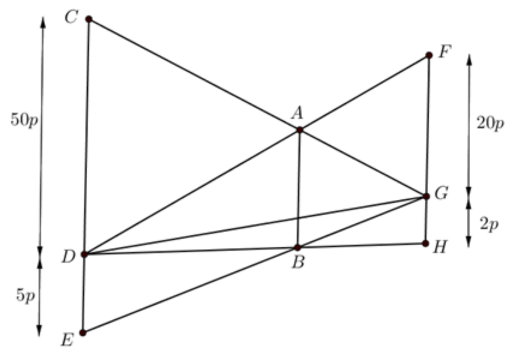


Some experimentation will hopefully convince us that the largest hexagon within a square looks like this. Now using the sine rule in the top left corner, we have

$$\frac{x - \frac{1}{\sqrt{2}}}{\sin 120^\circ} = \frac{1}{\sin 45^\circ}.$$

This tells us that $x = 1.93185...$, and so $\frac{\text{area of hexagon}}{\text{area of square}} = \frac{6 \times \frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2}}{1.93185^2} =$ 0.696. Thus our answer is 696.

18.



In this diagram, which is not to scale, the straight lines CE and FH are parallel. The length of CD is $50p$. The length of DE is $5p$. The length of FG is $20p$. The length of GH is $2p$. Find the length of AB to the nearest integer multiple of p. For example, if the exact length of AB was $5.72p$, the answer would be $6p$.

Answer to 18; Triangles CDA and GFA are similar, with $\dfrac{FG}{CD} = \dfrac{2}{5}$.

Triangles DBE and HBG are similar, with $\dfrac{GH}{DE} = \dfrac{2}{5}$.

So the heights of triangles CDA and GFA are in the ratio $\dfrac{2}{5}$, and the heights of triangles DBE and HBG are also in the ratio $\dfrac{2}{5}$.

Thus AB is parallel to CE and FH.

Triangles CGE and AGB are similar,and so $\dfrac{AB}{55p} = \dfrac{2}{7}$, and so $AB = \dfrac{110p}{7}$.

So $AB = 16p$ (to the nearest integer multiple of p).

19. Luke is working with logarithms to base 10. He makes three mistakes in a row; he says that

1. $\log(6) + \log(a) = \log(6 + a)$

2. $\log(b) - \log(6) = \log(b - 6)$

3. $\log(c^6) = (\log(c))^6$

Strangely, however, a, b and c are all numbers bigger than 1 such that the equations he's written down do in fact hold. What is the integer part of abc?

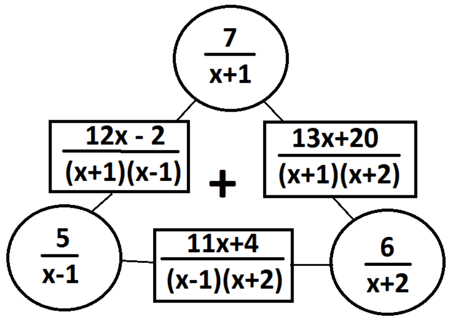
Answer to 19; we certainly know $\log(6) + \log(a) = \log(6a)$, so we must have $6 + a = 6a$, which gives us $a = 1.2$.

We also know that $\log(b) - \log(6) = \log(b/6)$, so we must have $\frac{b}{6} = b - 6$, and so $b = 7.2$.

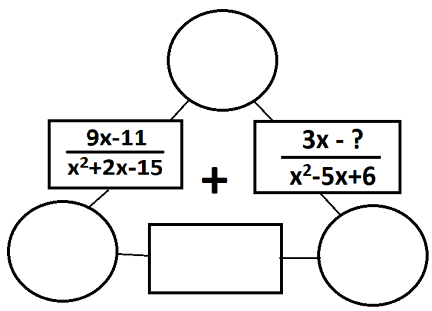
Finally we know $\log(c^6) = 6\log(c)$, and so (since $c \neq 1$) $6\log(c) = (\log(c))^6 \implies 6 = (\log(c))^5 \implies c = 10^{\sqrt[5]{6}} = 26.98$.

So $abc = 233.06....$ and the answer we need is 233.

20. A particular kind of arithmagon works like this;



Find the question-mark in this arithmagon;



Answer to 20; factorising the denominators gives $(x+5)(x-3)$ and $(x-3)(x-2)$, so the denominators of the circles must be $x-3, x-2$, and $x+5$ clockwise from the top (let's call the numerators a, b, and c).

Now looking at the left-hand side, we have $c(x-3)+a(x+5) = 9x-11$. Comparing the x-terms and the constant terms give simultaneous equations that solve to give $a = 2, c = 7$.

Now looking at the right-hand side, we have $b(x-3)+2(x-2) = 3x-?$. Comparing the x-terms and the constant terms give $b = 1, ? = 7$.