# Ritangle 2018 – Questions 1-25

## Answers in red

Q1. How many 8-digit numbers divisible by 18 are there such that every digit is either a 1 or a 2 or a 3?

Q1. Any such number must be divisible by 9 and by 2.

So it must be even and end in a 2.

The digits must also sum to a multiple of 9.

Seven 1s and a 2 add to 9, so 11111112 is one possible number.

Seven 3s and a 2 add to 23, so the only other possibility for the digit sum is 18.

We could arrive at this in three ways;

Four 3s, two 2s and two 1s, giving $\binom{7}{4}\binom{3}{2}$ possibilities.

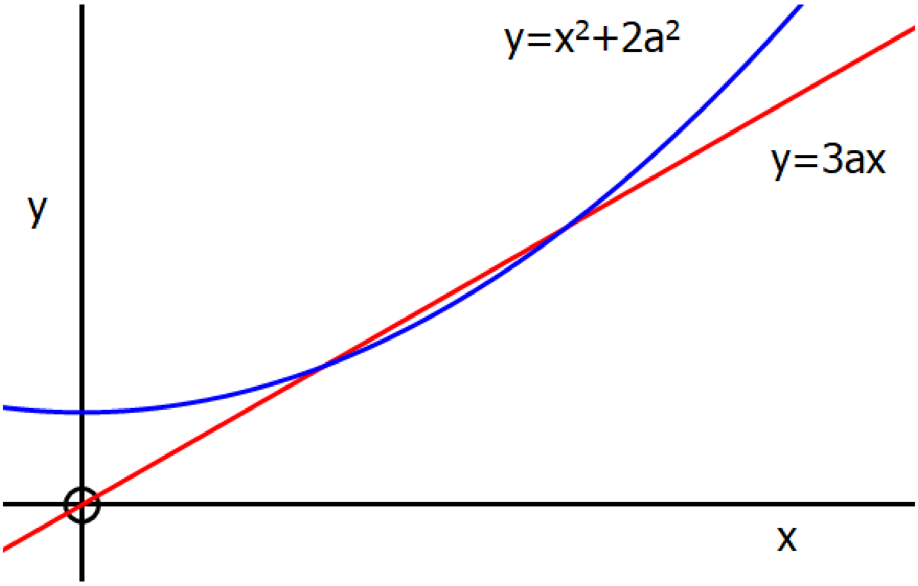
Three 3s, four 2s and a 1, giving $\binom{7}{3}\binom{4}{1}$ possibilities.

Two 3s and six 2s, giving $\binom{7}{2}$ possibilities.

Adding these up, we get 267 possible numbers.

Q2. The line y = 3ax (where a is positive) and the curve y = $x^2 + 2a^2$ enclose an area of size a. What is a?

Q2.




Where do y = $x^2+2a^2$ and y = 3ax cut? Solving $x^2+2a^2$ = 3ax gives x = a, 2a. So

$$a = \int_a^{2a} 3ax\,dx - \int_a^{2a} x^2 + 2a^2\,dx = \left[\frac{3ax^2}{2} - \frac{x^3}{3} - 2a^2x\right]_a^{2a}$$

$$= \left(6a^3 - \frac{8a^3}{3} - 4a^3\right) - \left(\frac{3a^3}{2} - \frac{a^3}{3} - 2a^3\right) = \frac{a^3}{6}$$

Solving gives a = $\sqrt{6}$ .

Q3. When the expression $10x^2 + 100x + 10$ is divided by x - a, the remainder is b, and when it is divided by x - b the remainder is a. Given that a $\neq$ b, find the remainder when the expression is divided by x – (a + b).

Q3. The remainder theorem gives us that

$$10a^2 + 100a + 10 = b, \quad 10b^2 + 100b + 10 = a.$$

Subtracting, we have $10(a^2 - b^2) + 100(a - b) = b - a$.

Dividing by a - b (since $a - b \neq 0$) gives

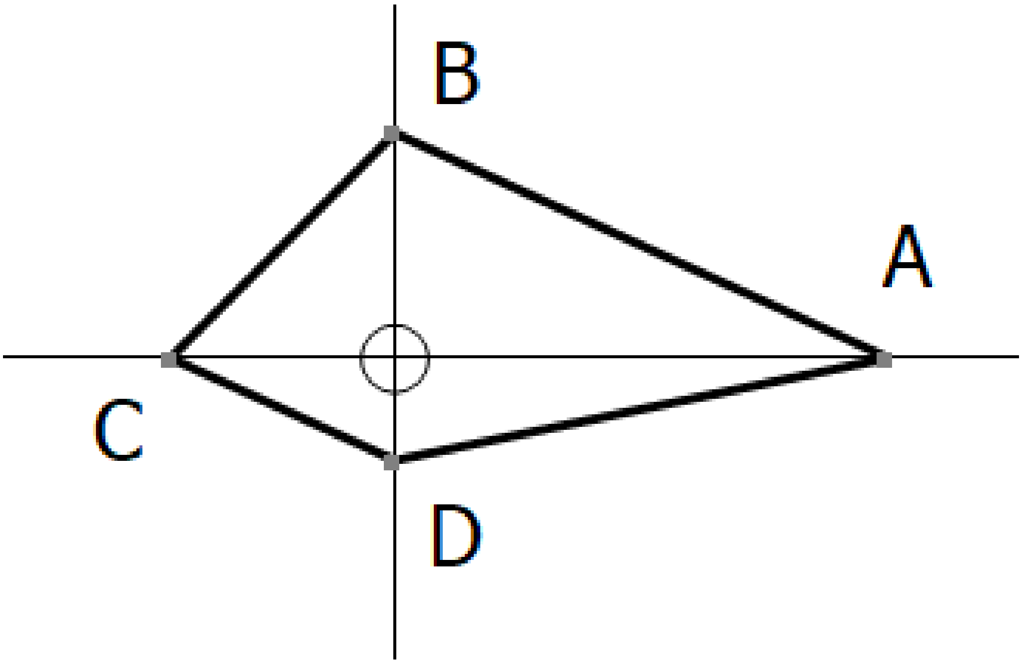
$$10(a + b) + 100 = -1, \text{ and so } a + b = -10.1.$$

Thus the remainder when we divide by a + b is

$$10(-10.1)^2 + 100(-10.1) + 10 = 20.1.$$

Q4. A quadrilateral is formed by the points A, B, C and D, where A is (a, 0), B = (0, b), C = (-1/b, 0), and D = (0, -1/a) and a and b are positive. Show that ABCD is always a trapezium, and find its area in terms of a and b. If a = 11, what value of b minimizes the area of the trapezium?

Q4.




Gradient of AB is $-\frac{b}{a}$, the gradient of CD is $-\frac{1/a}{1/b} = -\frac{b}{a}$, thus AB and CD are parallel.

Area = $\frac{1}{2}(ab+1+1+\frac{1}{ab}) = \frac{a^2b^2+2ab+1}{2ab} = \frac{(ab+1)^2}{2ab}$.

If a = 11, then area = $\frac{(11b+1)^2}{22b} = \frac{1}{22}\left(121b+22+\frac{1}{b}\right)$.

The numbers 121b and 1/b have a constant product, so 121b + 1/b is minimised when 121b = 1/b, which gives b = 1/11.

Q5. Angles are in radians. An infinite sequence $x_0, x_1, x_2, x_3, \ldots$ is defined like this;

$$x_0 = 1,\ x_{2n+1} = \cos(x_{2n}),\ x_{2n+2} = \arctan(x_{2n+1}),\ \text{for all } n \geq 0.$$

Find the limit to which the sequence $y_n = x_{2n+1} - x_{2n+2}$ converges.

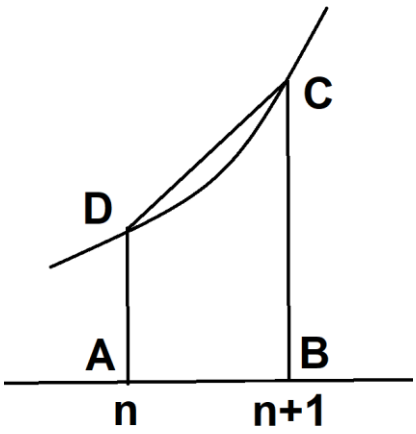
*A spreadsheet may help you in the above question.*

Q5. The spreadsheet below shows you what happens;

| ***Start*** | ***cos(left)*** | ***arctan(left)*** | |
|---|---|---|---|
| 1 | 0.540302306 | 0.495367289 | |
| 0.495367289 | 0.879794176 | 0.721538843 | |
| 0.721538843 | 0.75079015 | 0.644006613 | |
| 0.644006613 | 0.799696595 | 0.674555912 | |
| 0.674555912 | 0.780984384 | 0.663038028 | |
| 0.663038028 | 0.788125922 | 0.667458598 | |
| 0.667458598 | 0.785397308 | 0.665773221 | |
| 0.665773221 | 0.786439427 | 0.666417432 | |
| 0.666417432 | 0.786041356 | 0.666171432 | |
| 0.666171432 | 0.786193403 | 0.666265405 | |
| 0.666265405 | 0.786135326 | 0.666229512 | |
| 0.666229512 | 0.786157509 | 0.666243222 | |
| 0.666243222 | 0.786149036 | 0.666237985 | |
| 0.666237985 | 0.786152272 | 0.666239985 | |
| 0.666239985 | 0.786151036 | 0.666239221 | |
| 0.666239221 | 0.786151508 | 0.666239513 | |
| 0.666239513 | 0.786151328 | 0.666239402 | |
| 0.666239402 | 0.786151397 | 0.666239444 | |
| 0.666239444 | 0.78615137 | 0.666239428 | |
| 0.666239428 | 0.786151381 | 0.666239434 | |
| 0.666239434 | 0.786151377 | 0.666239432 | |
| 0.666239432 | 0.786151378 | 0.666239433 | |
| 0.666239433 | 0.786151378 | 0.666239432 | |
| 0.666239432 | 0.786151378 | 0.666239433 | |
| 0.666239433 | 0.786151378 | 0.666239432 | |
| 0.666239432 | 0.786151378 | 0.666239432 | |
| 0.666239432 | 0.786151378 | 0.666239432 | |
| 0.666239432 | 0.786151378 | 0.666239432 | 0.119911945 |

So to 4s.f. 0.1199 is the final answer.

Q6.

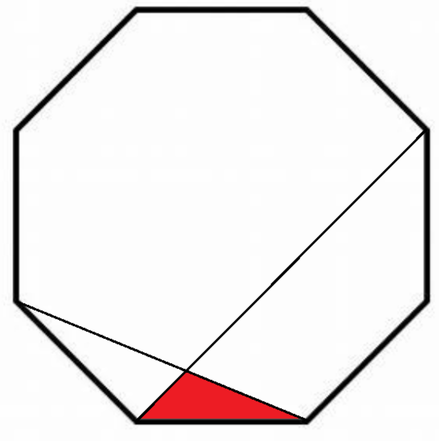



The area bounded by $y = x^2$, the x-axis and the lines $x = n$ and $x = n + 1$ is overestimated by the trapezium ABCD. Find the size of the overestimate.

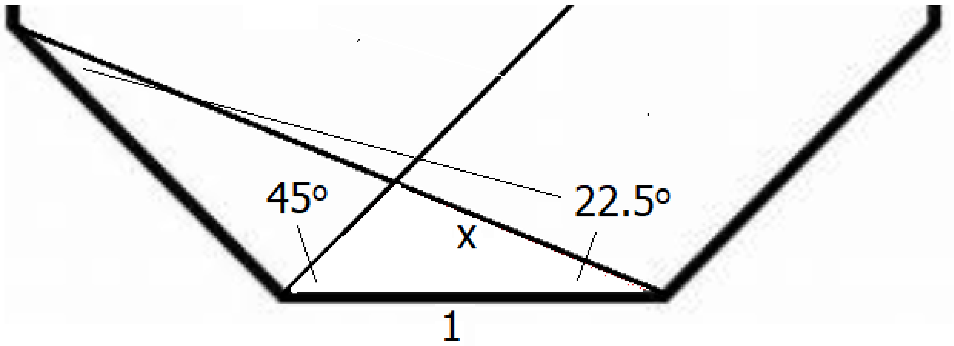
Q6. Overestimate = area of trapezium – area under curve =

$$\frac{(n^2+(n+1)^2)\times 1}{2}-\int_n^{n+1} x^2 dx = n^2+n+\frac{1}{2}-\left[\frac{x^3}{3}\right]_n^{n+1} = n^2+n+\frac{1}{2}-\frac{(n+1)^3}{3}+\frac{n^3}{3}=\frac{1}{6}$$

Q7. What percentage of the regular octagon is shaded?



Q7. Let the octagon have side 1, which means the area of the octagon is $2+2\sqrt{2}$.




From the sine rule, $\frac{x}{\sin 45^o} = \frac{1}{\sin 112.5^o} \Rightarrow x = \frac{\sin 45^o}{\sin 112.5^o}$.

The area of the triangle is therefore (using A = $\frac{1}{2}ab\sin C$)

$$\frac{1}{2}x\sin 22.5^o = \frac{\sin 22.5^o \sin 45^o}{2\sin 112.5^o} = 0.146.....$$

Thus the percentage shaded is $\frac{0.146...}{2+2\sqrt{2}} \times 100\% = 3.033008....$

So to 4s.f 3.033 is the final answer.

Q8. A biased six-sided dice showing the faces 1, 2, 3, 4, 5, 6 is rolled 21 times. One face shows once, another twice, a third three times, a fourth four times, a fifth five times and the sixth six times. If the median result is 3, the IQR is 4, and $\sum x = 80$, what is $\sum x^2$ ?

Q8. The only values that work are;

| Score | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Freq | 2 | 4 | 5 | 1 | 3 | 6 |

So this gives us $\sum x^2 = 370.$

Q9. The floor function $\lfloor x \rfloor$ is defined as the smallest integer less than or equal to x, while the ceiling function $\lceil x \rceil$ is defined as the smallest integer greater than or equal to x.

Thus $\lfloor 3 \rfloor = \lceil 3 \rceil = 3, \quad \lfloor 5.1 \rfloor = \lceil 4.9 \rceil = 5.$

Define a sequence $u_n = \left\lfloor \left(\frac{n}{10}\right)^2 \right\rfloor + \left\lceil \left(\frac{n}{10}\right)^2 \right\rceil$ for n > 0. What is the smallest value of n so that $u_{n+1} = u_n + 4$?

*A spreadsheet may help you in the above question.*

Q9. Using a spreadsheet here gives;

| *n* | *(n/10)^2* | *floor((n/10)^2)* | *ceiling ((n/10)^2)* | *ceiling + floor* | *u_n+1-u_n* |
|---|---|---|---|---|---|
| 1 | 0.01 | 0 | 1 | 1 | x |
| 2 | 0.04 | 0 | 1 | 1 | 0 |
| 3 | 0.09 | 0 | 1 | 1 | 0 |
| 4 | 0.16 | 0 | 1 | 1 | 0 |
| 5 | 0.25 | 0 | 1 | 1 | 0 |
| 6 | 0.36 | 0 | 1 | 1 | 0 |
| 7 | 0.49 | 0 | 1 | 1 | 0 |
| 8 | 0.64 | 0 | 1 | 1 | 0 |
| 9 | 0.81 | 0 | 1 | 1 | 0 |
| 10 | 1 | 1 | 1 | 2 | 1 |
| 11 | 1.21 | 1 | 2 | 3 | 1 |
| 12 | 1.44 | 1 | 2 | 3 | 0 |
| 13 | 1.69 | 1 | 2 | 3 | 0 |
| 14 | 1.96 | 1 | 2 | 3 | 0 |
| 60 | 36 | 36 | 36 | 72 | 3 |
| 61 | 37.21 | 37 | 38 | 75 | 3 |
| 62 | 38.44 | 38 | 39 | 77 | 2 |
| 63 | 39.69 | 39 | 40 | 79 | 2 |
| 64 | 40.96 | 40 | 41 | 81 | 2 |
| 65 | 42.25 | 42 | 43 | 85 | 4 |
| 66 | 43.56 | 43 | 44 | 87 | 2 |
| 67 | 44.89 | 44 | 45 | 89 | 2 |
| 68 | 46.24 | 46 | 47 | 93 | 4 |
| 69 | 47.61 | 47 | 48 | 95 | 2 |
| 70 | 49 | 49 | 49 | 98 | 3 |
| 71 | 50.41 | 50 | 51 | 101 | 3 |
| 72 | 51.84 | 51 | 52 | 103 | 2 |

Thus 64 is the value required.

Q10. Two numbers x and y are such that 0 < x, y < 1. The sum to infinity of the geometric series with first term x and common ratio y is 2. The sum to infinity of the geometric series with first term y and common ratio x is 3. What is xy?

Q10. We know $S_\infty = \dfrac{a}{1-r}$ . Thus

$$\frac{x}{1-y} = 2, \frac{y}{1-x} = 3 \Rightarrow \frac{x}{1-3(1-x)} = 2 \Rightarrow x = \frac{4}{5}, y = \frac{3}{5}$$

Thus $xy = \frac{12}{25}$.

Q11. A palindromic number is one that is equal to its reflection (it reads the same forwards and backwards). It's recently been shown that every positive integer is the sum of three positive palindromic numbers. For example,

$$2587876=2534352+18981+34543.$$

If you are given that

$$652641310=1\_5\_1+34\_6\_43+649\_4\_946,$$

where the three numbers on the right are palindromic, find the six digits that fill the gaps. What is the product of these six digits?

Q11. We have, where x, y and z are the missing digits,

$$652641310 = 10501 + 1010x + 3406043+10100y + 649040946+z101000.$$

Thus $183820=1010(100z+10y+x)$, and so $100z+10y+x=182$

Thus $z = 1$, $y = 8$, $x = 2$, and the product of the six digits is 256.

**Q12.** What number do you get if the number of distinct arrangements of the letters in the string HUBBAHUBBA is divided by the number of distinct arrangements of the letters in the string HUBBA?

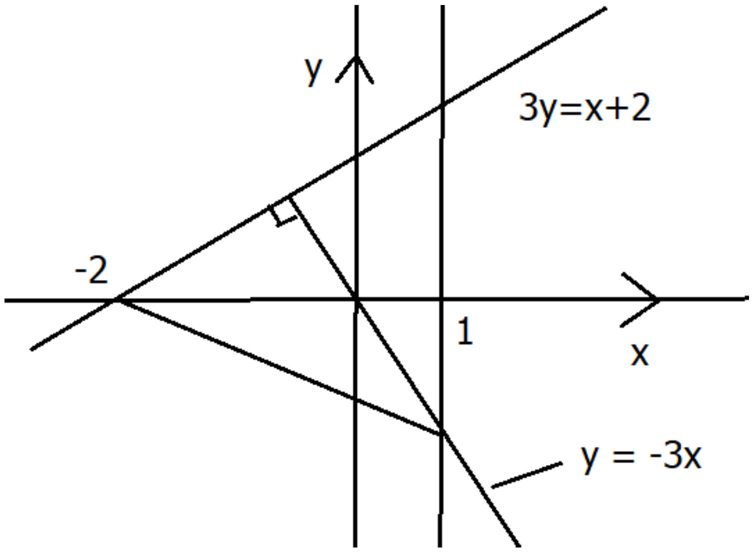
Q12.

$$\frac{10!/(2!2!2!4!)}{5!/2!} = 315.$$

Q13. Given a triangle ABC, try dropping a perpendicular from A to BC, and another from B to AC and a third from C to AB (these lines are called the ***altitudes*** of the triangle). You should find that they meet, whatever triangle you have chosen, in a point that is called the ***orthocentre***.

A triangle has its orthocentre at the origin. One of its sides is part of the line 3y = x + 2, while another side is part of the line x = 1. Find the perimeter of the triangle.

Q13.




The three vertices of the triangle are (1,1), (1,-3) and (-2,0).

Thus its perimeter is $4+\sqrt{3^2+3^2}+\sqrt{3^2+1^2}=4+3\sqrt{2}+\sqrt{10}$

Q14. What is the value of the term in the expansion of $\left(6x^3 - \frac{5}{x^2}\right)^{10}$ that is independent of x?

Q14. If we expand the bracket, we get the sum of terms of the form

$$\binom{10}{r}(6x^3)^r\left(-\frac{5}{x^2}\right)^{10-r}$$

For this to be independent of x, 3r=2(10-r), which gives r = 4.

So this term becomes

$$\binom{10}{4}(6)^4(-5)^6 = 4252500000.$$

Q15. If y = 30sin(40x+72º)+40cos(72x+30º)+72tan(30x+40º), what, in degrees, is the period of the function y?

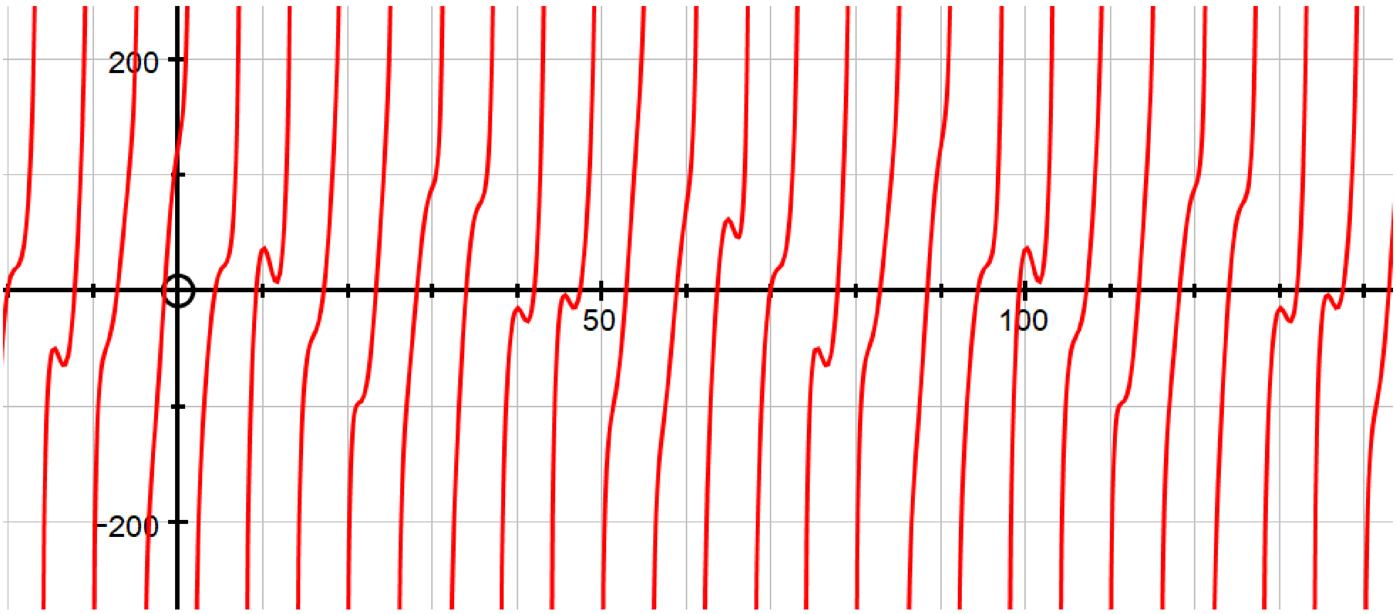
Q15. If y = asin(bx + c), then the period is affected by b, but not by a or c. (The period here is 360º/b.) The same is true for cos and tan (although the period for tan is 180º.)

Thus the period of 30sin(40x + 72º) is 360º/40 = 9º, and

The period of 40cos(72x + 30º) is 360º/72 = 5º, and

The period of 72tan(30x + 40º) is 180º/30 = 6º.

So we need the LCM of 9, 5, and 6, which is 90. The function y has a period of 90º.



Q16. You are given that $a^b c^d = 46656$, where $a \leq c$ and a, b, c and d are all positive integers, with a, b, c, d $\geq 2$.

How many possibilities for (a, b, c, d) are there?

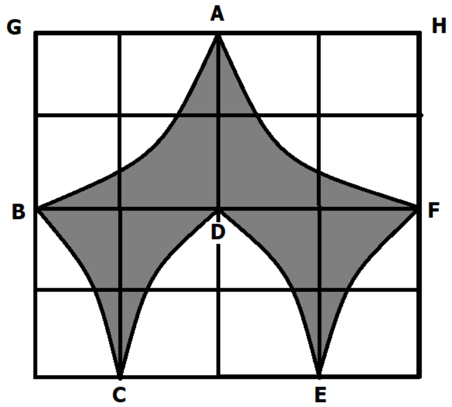
Q16. We see that $46656 = 2^6 3^6$, so b and d can take the values 2, 3, 4, 6.

| (a,c) | 2=d | 3=d | 4=d | 6=d |
|---|---|---|---|---|
| 2=b | $(2,2^2 3^3)$<br>$(3,3^2 2^3)$<br>$(2^2,2^1 3^3)$<br>$(2^1 3^1,2^2 3^2)$<br>$(2^3,3^3)$<br>$(3^2,2^3 3)$<br>$(2^2 3,2^1 3^2)$ | $(2^3,3^2)$ | $(2^1 3^1,2^1 3^1)$ | x |
| 3=b | $(2^2, 3^3)$ | $(2, 2^1 3^2)$<br>$(2^1 3^1,2^1 3^1)$<br>$(3,2^2 3)$<br>$(2^2,3^2)$ | x | x |
| 4=b | $(2,2^1 3^3)$<br>$(3,2^3 3)$ | | x | x |
| 6=b | $(2,3^3)$<br>$(3,2^3)$ | $(2,3^2)$<br>$(3,2^2)$ | x | (2,3) |

So there are 21 possibilities.

Q17. The Franciscan church in Nice looks like this;

Suppose a nearby church includes this in its architecture (the picture is drawn only roughly);




The grid is comprised of sixteen 1 by 1 squares.

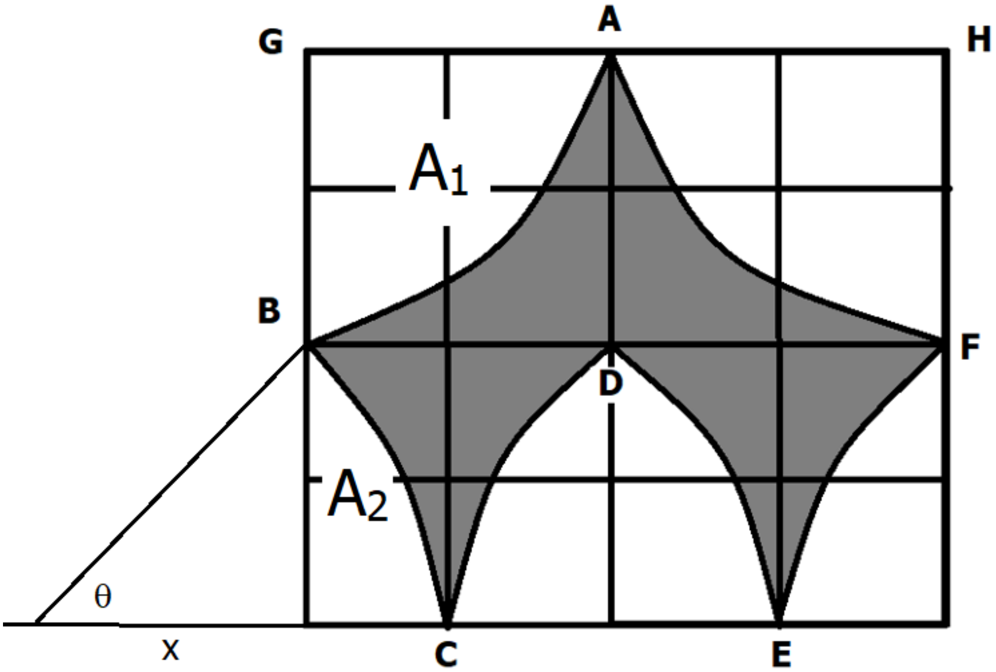
The curves AB and AF are arcs from circles centred at G and H respectively.

The curves BC, CD, DE and EF are all arcs from circles with their centres on the straight line that includes C and E.

What is the shaded area?

Q17. The shaded area is $16 - 2A_1 - 4A_2$, where

$$A_1 = \frac{4\pi}{4} = \pi.$$




Let the radius of the circle containing the arc BC be r.

Then $r = x+1$, and $r^2=x^2+4$.

Thus $(1+x)^2 = x^2 + 4$ and $x = 1.5$, and $\tan\theta = 4/3$.

Now $A_2$ = $\frac{1}{2}\arctan(4/3)(2.5)^2 - \frac{1}{2}2(1.5) = 1.397\ldots$

So shaded area is 4.125624469…

So to 4s.f. 4.126 is the final answer.

Q18. You are given that $f(x) = ax^3 + bx^2 + cx + d$ .

You are also told that $f(0) = 0, f'(1) = 1, f''(2) = 2, f'''(3) = 3.$

What is $(c+d)^{(a+b)}$ ?

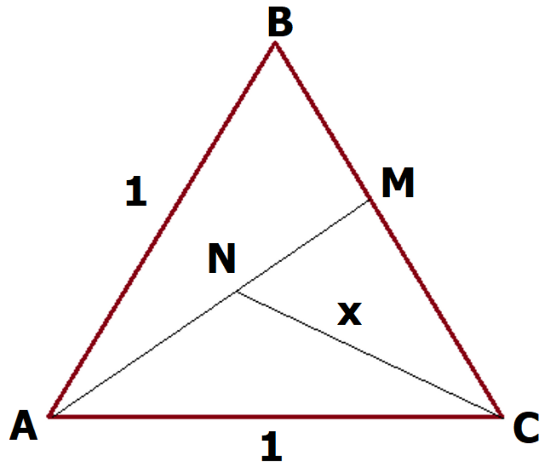
Q18. f'(x) = $3ax^2+2bx+c$, f''(x) = 6ax+2b, f'''(x) = 6a.

So 3=6a, a = 0.5, and 2 = 6 + 2b, b = -2, and 1 = 1.5 - 4 + c, c=3.5, d = 0.

Thus $(c+d)^{(a+b)}=3.5^{-1.5}$=0.1527207097…

So to 4s.f. 0.1527 is the answer.

Q19. An equilateral triangle ABC of side 1 is divided into three equal areas by two lines AM and CN as in the diagram below. What is the length x?




Q19. 2 x Area ABM = Area AMC, so BM = 1/3, MC = 2/3.

Similarly AN = NM = y.

Cos rule in ABM; $4y^2 = 1 + 1/9 \; -2\frac{1}{3}\cos\frac{\pi}{3}$ , so $y = \frac{\sqrt{7}}{6}$.

Sin rule in ABM; $\frac{\sin(AMB)}{1} = \frac{\sqrt{3}/2}{2\sqrt{7}/6} = \frac{3\sqrt{3}}{2\sqrt{7}}$ = sin AMC.

Thus cos AMC = $\sqrt{1-\left(\frac{27}{28}\right)} = \frac{1}{\sqrt{28}}$ .

Cos rule in NMC; $x^2 = y^2 + 4/9 - \; 2y\frac{2}{3}\sqrt{\frac{1}{28}}$ , so x = 0.7264831573…

So to 4s.f. 0.7265 is the final answer.

Q20.

Find $\sum_{1}^{100}(n^2\int_{\sqrt[n]{n}}^{\sqrt[n]{n+1}} x^{n-1}dx)$.

Q20.

$$\sum_{1}^{100}(n^2\int_{\sqrt[n]{n}}^{\sqrt[n]{n+1}} x^{n-1}dx)=\sum_{1}^{100}\left(n^2\left[\frac{x^n}{n}\right]_{\sqrt[n]{n}}^{\sqrt[n]{n+1}}\right)$$

$$=\sum_{1}^{100}\Big(n\big((n+1)-(n)\big)\Big)=\sum_{1}^{100}n=5050.$$

Q21. You are given a square of side x with perimeter P and area A, and a rectangle with sides x and y, where y does not equal x, which has perimeter P' and area A'. The numerical values P', P, A', A are four consecutive terms from an arithmetic sequence. What is the numerical value of P + A + P'+ A'?

Q21. P' = 2x + 2y, P = 4x, A' = xy, A = $x^2$.

So we have 2x+2y +d = 4x, 4x+d = xy, xy + d = $x^2$.

So d = 2x-2y, yielding 4x +(2x-2y) = xy, xy + (2x-2y) = $x^2$.

So we have $y = \frac{6x}{x+2} \Rightarrow x\frac{6x}{x+2} + 2x - 2\frac{6x}{x+2} = x^2.$

Solving this gives x = 0, y = 0 (not allowed), x = 4,y = 4 (not allowed), and

x = 2, y = 3. So our four terms are P' = 10, P = 8, A' = 6, A = 4.

The sum of these values is 28.

Q22. You are given that

$$\frac{a+\sqrt{b}}{5-3\sqrt{b}} = c + d\sqrt{b},$$

where a, b, c, d are integers so that 0 < a, b, c, d < 7 and b is not a square. Find a×b×c×d.

Q22. Multiplying out, $a+\sqrt{b} = (5c-3db)+(5d-3c)\sqrt{b}$.

Since b is not a square, $\sqrt{b}$ is irrational, so for this equation to work,

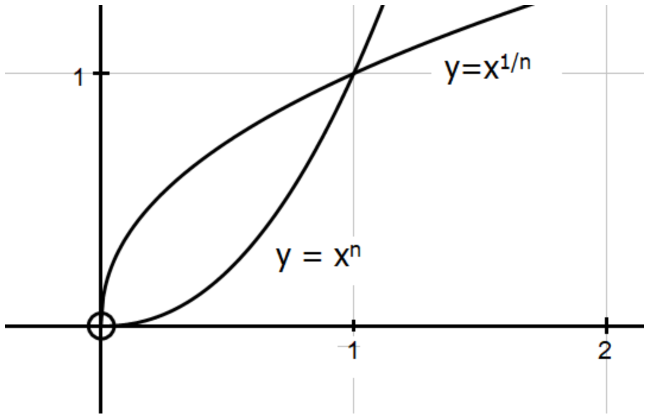
a = 5c-3db, 5d-3c = 1. If 0 < c, d < 7, there is only one solutions to 5d-3c=1, that is, d = 2, c = 3.

So we also need to solve a = 15 – 6b, or a + 6b = 15, and a = 3, b = 2 is the only solution in our range.

Thus a = 3, b = 2, c = 3, d = 2, and the product of these is 36.

Q23. The area enclosed between the curves y = $x^n$ and y = $x^{1/n}$ , where n > 1, is 1/100. What is n?

Q23.




Enclosed area =

$$\int_0^1 x^{1/n} - x^n dx = \left[ \frac{x^{n+1/n}}{n+1/n} - \frac{x^{n+1}}{n+1} \right]_0^1$$

$$= \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1} = \frac{1}{100} \Rightarrow n = \frac{101}{99.}$$

Q24. The three angles of a triangle (measured in degrees) are three consecutive terms from a geometric sequence. The sizes of the three angles multiply together to give 20. What, in degrees, is the smallest angle?

Q24. Say the three angles in degrees are a, ar and $ar^2$.

So $a + ar + ar^2 = 180$, and $a(ar)(ar^2)=20 = (ar)^3$.

Thus $ar = \sqrt[3]{20} = k \Rightarrow r = \dfrac{k}{a}$ . So

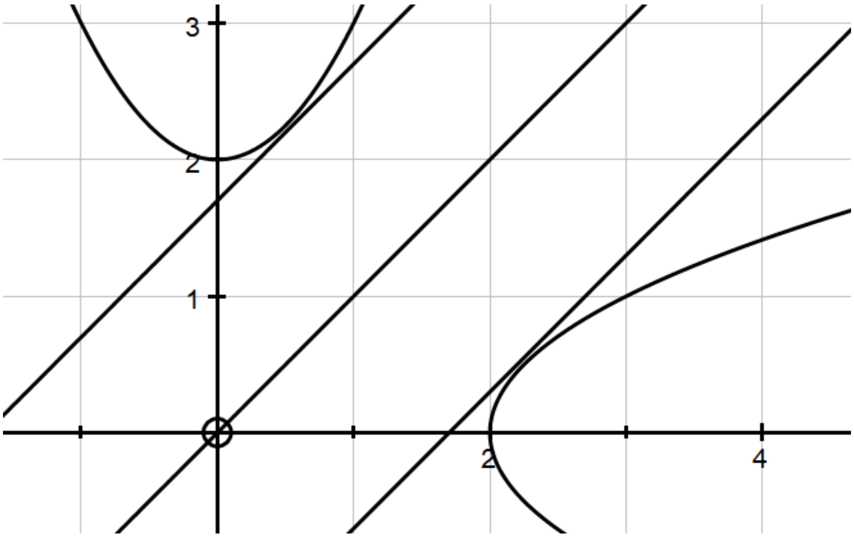
$$a + k + \frac{k^2}{a} = 180 \Rightarrow a^2 + (k-180)a + k^2 = 0$$
$$\Rightarrow a = \frac{180 - k \pm \sqrt{(180-k)^2 - 4k^2}}{2}$$

Substituting in for k gives a = 177.24…. or 0.0415701659… (in degrees).

We want the smaller of these, so to 4s.f. the answer is a = 0.04157.

Q25. The point A is on the parabola $y = x^2+2$, while B is a point on the parabola $x = y^2+2$. What is the smallest that the distance AB can be?

Q25.




The parabola $y = x^2 + 2$ is the reflection of the parabola $x = y^2 + 2$ in $y = x$.

This means the smallest AB can be is when the gradient of the curves at both A and B are 1.

At A, $y' = 2x = 1$, so $x = 0.5$, $y = 2.25$, so B is (2.25,0.5).

Thus the distance AB is $\sqrt{2(2.25-0.5)^2} = 1.75\sqrt{2} = 2.474873734...$

So 2.475 is the answer to 4s.f.